CHARACTER POLYNOMIALS AND KRONECKER PRODUCTS

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Menu

- **Kronecker Products of representations of S_n**
- **Characters Polynomials**
- **Duality with conjugacy class products**

- **The kronecker product is a tensor product of linear operators :**

$$T_1 : V \to V \; :$$
$$T_2 : W \to W$$

$$T_1 \otimes T_2 : V \otimes W \to V \otimes W$$
$$v_i \otimes w_j \mapsto T_1(v_i) \otimes T_2(w_j)$$

- (Inner) **Tensor product of representations of S_n**

$$A : S_n \to Aut(V) \quad B : S_n \to Aut(W)$$
$$\sigma \mapsto A(\sigma) \qquad \sigma \mapsto B(\sigma)$$

$$A \otimes B : S_n \to Aut(V \otimes W)$$
$$\sigma \mapsto A(\sigma) \otimes B(\sigma)$$

- **Definition**. $A \otimes B$ is called the Kronecker Product of A and B.

Irreducible Representations of S_n

- Irreducible representations of S_n are indexed with partitions λ of n :

$$A^\lambda : S_n \to Aut(V)$$
$$\sigma \mapsto A^\lambda(\sigma)$$

- Character of a tensor product $A^\lambda \otimes A^\mu$

$$\chi^{A^\lambda \otimes A^\mu} : S_n \to Q$$
$$\sigma \mapsto tr(A^\lambda \otimes A^\mu)(\sigma)$$
$$= tr(A^\lambda)(\sigma) \bullet tr(A^\mu)(\sigma)$$

$$\therefore \quad \chi^{A^\lambda \otimes A^\mu}(\sigma) = \chi^\lambda(\sigma) \bullet \chi^\mu(\sigma)$$

- Characters are constant functions on Conjugacy classes :

$$\sigma_1 \in C_\mu \text{ et } \sigma_2 \in C_\mu \quad \Rightarrow \quad \chi^\lambda(\sigma_1) = \chi^\lambda(\sigma_2) := \chi^\lambda_\mu$$

$\left[\chi^\lambda_\mu\right]_{\lambda,\mu}$:= Characters Table of S_n.

● Exemple 1. Character table [χ^{λ}_{μ}] of S_3

$\lambda \setminus \mu$	(3)	(2,1)	(1,1,1)
(3)	1	1	1
(2,1)	–1	0	2
(1,1,1)	1	–1	1

$$\chi^{(2,1)\otimes(2,1)} = (1,0,4)$$
$$= (1,1,1)+(-1,0,2)+(1,-1,1)$$
$$= \chi^{(3)} + \chi^{(2,1)} + \chi^{(1,1,1)}$$

● The problem of the Kronecker product: Decompose

$$\chi^{\lambda} \otimes \chi^{\mu} = \sum_{\alpha \vdash n} t^{\alpha}_{\lambda,\mu} \chi^{\alpha}$$

An algebraic answer to the problem :

$$\chi^{\lambda} \otimes \chi^{\mu} |_{\chi^{\alpha}} = t^{\alpha}_{\lambda,\mu} = \sum_{\gamma \vdash n} \frac{|C_{\gamma}|}{n!} \chi^{\lambda}_{\gamma} \chi^{\mu}_{\gamma} \chi^{\alpha}_{\gamma}$$

This answer is not satisfactory combinatorially

Characters Polynomials

-Defined by Specht in 1960,
-Were given little attention (Kerber gave a table, Macdonald has an exercice)

Definition. For each partition λ of $k < n$, there exists a unique polynomial $q_\lambda(x_1,\ldots,x_n) \in Q[x_1, x_2, \ldots]$ called characters polynomial such that for each permutation $\sigma \in S_n$ of cycle type $\mu = 1^{m_1} \cdots n^{m_n}$

$$q_\lambda(m_1,\ldots,m_n) = \chi^{(n-|\lambda|,\lambda)}_{1^{m_1}\ldots n^{m_n}}$$

•Examples.

1-(well known) :

$$\chi_\mu^{(n-1,1)} = \text{number of fixed points minus one of } C_\mu$$
$$= m_1\text{-}1 \quad \text{if} \quad \mu = 1^{m_1} 2^{m_2} \cdots n^{m_n}$$

$$\Rightarrow \quad q_{(1)}(\boldsymbol{x}) = x_1 - 1$$

2- $$\chi_\mu^{(n-2,2)} = m_2 + \binom{m_1}{2} - m_1$$

$$\Rightarrow \quad q_{(2)}(\boldsymbol{x}) = x_2 + \binom{x_1}{2} - x_1$$

3- $$\chi_\mu^{(n-2,1^2)} = -m_2 + \binom{m_1 - 1}{2}$$

$$\Rightarrow \quad q_{(1,1)}(\boldsymbol{x}) = -x_2 + (x_1-1)(x_1-2)/2$$

Observation (Littlewood). The value of a character χ^{λ} on a class $C\mu$ does not depend on the first part λ_1 of λ.

Consequence of the definition :

$$q_\lambda q_\mu = \sum_\alpha t^{\alpha}_{\lambda,\mu} q_\alpha$$

The products of characters polynomials decomposes as the Kronecker Products !

Example 1(continued) :

$$\chi^{(n-1,1)} \otimes \chi^{(n-1,1)} = \chi^{(n-2,2)} + \chi^{(n-2,1^2)} + \chi^{(n-1,1)} + \chi^{(n)}$$

$$q_1 \cdot q_1 = q_{(2)} + q_{(1,1)} + q_{(1)} + q_{(0)}$$

$$(x_1 - 1)^2 = [x_2 + \frac{x_1(x_1-3)}{2}] + [-x_2 + \binom{x_1 - 1}{2}] + [x_1 - 1] + [1]$$

Question.

How to compute the characters polynomials $q_\lambda(\boldsymbol{x})$?

First recipe. **Use the umbral operators of Rota.**

a) Write Schur function s_λ in the basis $\{p_\mu\}_{\mu|-n}$

b) Replace p_i by $(ix_i - 1)$ in each

$$p_\mu = (p_1)^{m_1}(p_2)^{m_2}$$

c) Expand $\prod_{i\geq 1}(ix_i - 1)^{m_i}$ as a sum $\sum_\theta c_\theta \prod_i x_i^{\theta_i}$

d) Replace each $x_i^{\theta_i}$ by $(x_i)_{\theta_i}$ (umbral operator)

We obtain $q_\lambda(\boldsymbol{x})$

- Example. $q_{(3)}(\boldsymbol{x})$

a) $$s_{(3)} = \frac{1}{6}(p_{1^3} + 3p_{21} + 2p_3)$$

b) $$\frac{1}{6}((x_1 - 1)^3 + 3(2x_2 - 1)(x_1 - 1) + 2(3x_3 - 1))$$

c) $$\frac{1}{6}(x_1^3 - 3x_1^2 + 6x_1x_2 - 6x_2 + 6x_3)$$

d) $$q_{(3)}(\boldsymbol{x}) = \frac{1}{6}((x_1)_3 - 3(x_1)_2 + 6x_1x_2 - 6x_2 + 6x_3)$$

<u>Second Recipe</u>. Use recursive calculus à la Murnaghan-Nakayama :

a) Compute $q_\lambda(x_1,0,\ldots,0) = f^{(x_1-|\lambda|,\lambda)}$ as a polynomial in x_1.

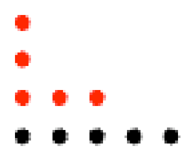
b) To obtain the terms containing $\binom{x_i}{j}$ and no variable x_k, with index $k > i$:

$$\binom{x_i}{j} \sum_{\substack{S=(\lambda=\lambda^0,\lambda^1,\ldots,\lambda^j),\\ |\lambda^r-\lambda^{r+1}|=i}} (-1)^{ht(S)} q_{\lambda^j}(x_1,\ldots,x_{i-1},0,\ldots)$$

where each λ^r-λ^{r+1} is a border strip of weight i and

$$ht(S) = \prod_{r=0}^{j-1}(height(\lambda^r - \lambda^{r+1}) - 1)$$

Example. $q_{(3,1,1)}(x_1, x_2,\ldots)$:



$$f^{(n-5,3,1,1)} + \binom{x_5}{1} - 2\binom{x_2}{2} q_{(1)}(\boldsymbol{x}) + \binom{x_2}{1} q_{(1,1,1)}(\boldsymbol{x}) - \binom{x_2}{1} q_{(3)}(\boldsymbol{x})$$

$$=$$

$$\frac{x_1!}{(x_1-2)(x_1-5)(x_1-6)(x_1-8)!\,20} + x_5 - 2\binom{x_2}{2}(x_1-1)$$

$$+\ x_2\left[\binom{x_1-1}{3} - \left(\binom{x_1}{3} - \binom{x_1}{2}\right)\right]$$

<u>Proofs</u>. Symmetric functions, plethystic substitution…

The Algebra of Characters polynomials

- The set $\{q_\lambda(x_1, x_2, \ldots)\}_\lambda$ of characters polynomials forms a linear basis of $Q[x_1, x_2, \ldots] = Q[\boldsymbol{x}]$

- With the following scalar product in $Q[x_1, x_2, \ldots]$:

$$\left\langle f, g \right\rangle_{Q[\boldsymbol{x}]} = \sum_{\substack{\alpha \vdash n \\ \alpha = 1^{a_1} 2^{a_2} \cdots n^{a_n}}} \frac{f(a_1, \ldots a_n) g(a_1, \ldots a_n)}{1^{a_1} \cdots n^{a_n} a_1! a_2! \cdots a_n!}, \; n \text{ large enough}$$

The basis $\{q_\lambda(\boldsymbol{x})\}_\lambda$ becomes orthonormal.

$\Rightarrow$ We can use the scalar product $\left\langle \,,\, \right\rangle_{Q[\boldsymbol{x}]}$ to compute the expansion of any polynomial $f \in Q[\boldsymbol{x}]$

$$f = \sum_\lambda c_\lambda q_\lambda(\boldsymbol{x}) \;\Rightarrow\; c_\lambda = \left\langle f, q_\lambda(\boldsymbol{x}) \right\rangle_{Q[\boldsymbol{x}]}$$

- Using truncated partitions, we can define the projective limit $Z = \left\langle \chi^\lambda \right\rangle$ of the centers $Z_n = \left\langle \chi^{n-|\lambda|, \lambda} \right\rangle$ of the group algebras of the symmetric groups S_n.

- In retrospective, the application

$$q : (Z, +, \otimes) \to (Q[\boldsymbol{x}], +, \bullet), \; q(\chi^\lambda) = q_\lambda(\boldsymbol{x})$$

is an algebra isomorphism and an isometry.

Remark. We have also defined and computed characters polynomials for Hecke Algebras of S_n.

Class Polynomials
(studied by A. Goupil, G. Schaeffer, S. Corteel)

Product of conjugacy classes of S_n

Example.

$$C(1^{n-2},2)*C(1^{n-3},3) = 4C(1^{n-4},4)+C(1^{n-5},3,2)+2(n-2)C(1^{n-2},2)$$

There exists a family $\{\omega_\mu\}_\mu$ of symmetric polynomials that multiply like conjugacy classes multiply in the group algebra.

Examples (Frobenius,1901 and Ingram, 1950)

$\omega_{(1^{n-2},2)} = p_1(\boldsymbol{x})$

$\omega_{(1^{n-3},3)} = p_2(\boldsymbol{x}) - \binom{n}{2}$

$\omega_{(1^{n-4},4)} = p_3(\boldsymbol{x}) - (2n-3)p_1(\boldsymbol{x})$

$\omega_{(1^{n-5},3,2)} = p_{(2,1)}(\boldsymbol{x}) - 4p_3(\boldsymbol{x}) - (\binom{n}{2} - 6n+8)p_1(\boldsymbol{x})$

$\omega_{(1^{n-2},2)}\,\omega_{(1^{n-3},3)} = 4\,\omega_{(1^{n-4},4)} + \omega_{(1^{n-5},3,2)} + 2(n-2)\,\omega_{(1^{n-2},2)}$

Fundamental Property of the polynomials $\omega_\mu(x)$:
If we evaluate the polynomials $\omega_\mu(x)$ on the contents $C(\lambda)=\{(j-i)\}_{(i,j)\in\lambda}$ of a Ferrers diagram λ, we obtain

$$\omega_\mu(C(\lambda)) = \frac{|C_\mu(n)|}{f^\lambda}\chi_\mu^\lambda$$

- There is no simple recipe for the production of the $\omega_\mu(x)$.

Class Polynomials $\omega_\mu(\boldsymbol{x})$
vs
Characters Polynomials $q_\mu(\boldsymbol{x})$

- Can we establish a combinatorial link between these two families of polynomials and deduce a combinatorial link between products of classes and Kronecker products ?