

Cluster complexes of bordered surfaces

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Main reference

S.F., Michael Shapiro, and Dylan Thurston,
Cluster algebras and triangulated surfaces.
Part I: Cluster complexes, [math.RA/0608367](#)

Plan

1. Arc complexes
2. Cluster complexes
3. Main results

1. Arc complexes

(after W.Thurston, J.Harer, R.Penner, . . .)

Bordered surfaces with marked points

Let S be a connected oriented surface with boundary. Fix a finite non-empty set M of *marked points* in the closure of S . Marked points in the interior of S are called *punctures*.

We will work with triangulations of S whose vertices are located at the marked points in M . We assume that M is non-empty, that there is ≥ 1 marked point on each boundary component, and that (S, M) is none of the following:

- a sphere with ≤ 3 punctures;
- a monogon with ≤ 1 punctures;
- an unpunctured digon or triangle.

Arcs

An *arc* γ in (S, M) is a curve in S , considered up to isotopy rel M , such that

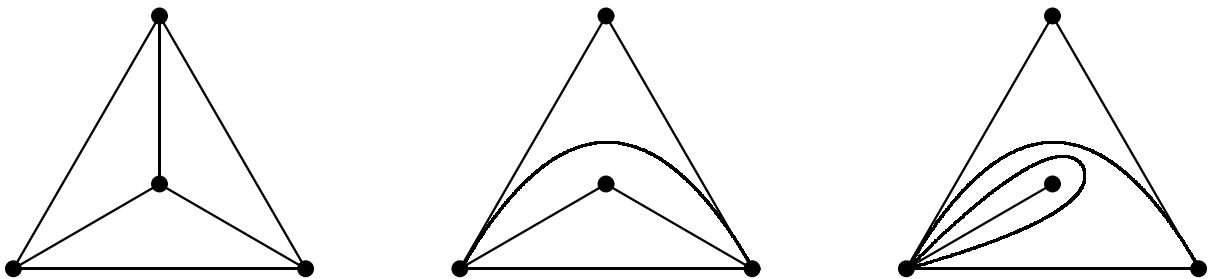
- γ connects marked points in M ;
- γ does not intersect itself except that its endpoints may coincide;
- the relative interior of γ is disjoint from M and from the boundary of S ;
- γ does not cut out an unpunctured monogon, or an unpunctured digon.

An arc whose endpoints coincide is called a *loop*.

Ideal triangulations

Two arcs are *compatible* if they do not intersect in the interior of S ; more precisely, there are curves in respective isotopy classes which do not intersect.

A maximal collection T of distinct pairwise compatible arcs is called an *ideal triangulation*. The arcs of T cut S into *ideal triangles*. Some of these triangles may be *self-folded*.



Each ideal triangulation consists of

$$n = 6g + 3b + 3p + c - 6$$

arcs, where g is the genus of S , b is the number of boundary components, p is the number of punctures, and c is the number of marked points on the boundary.

Arc complex

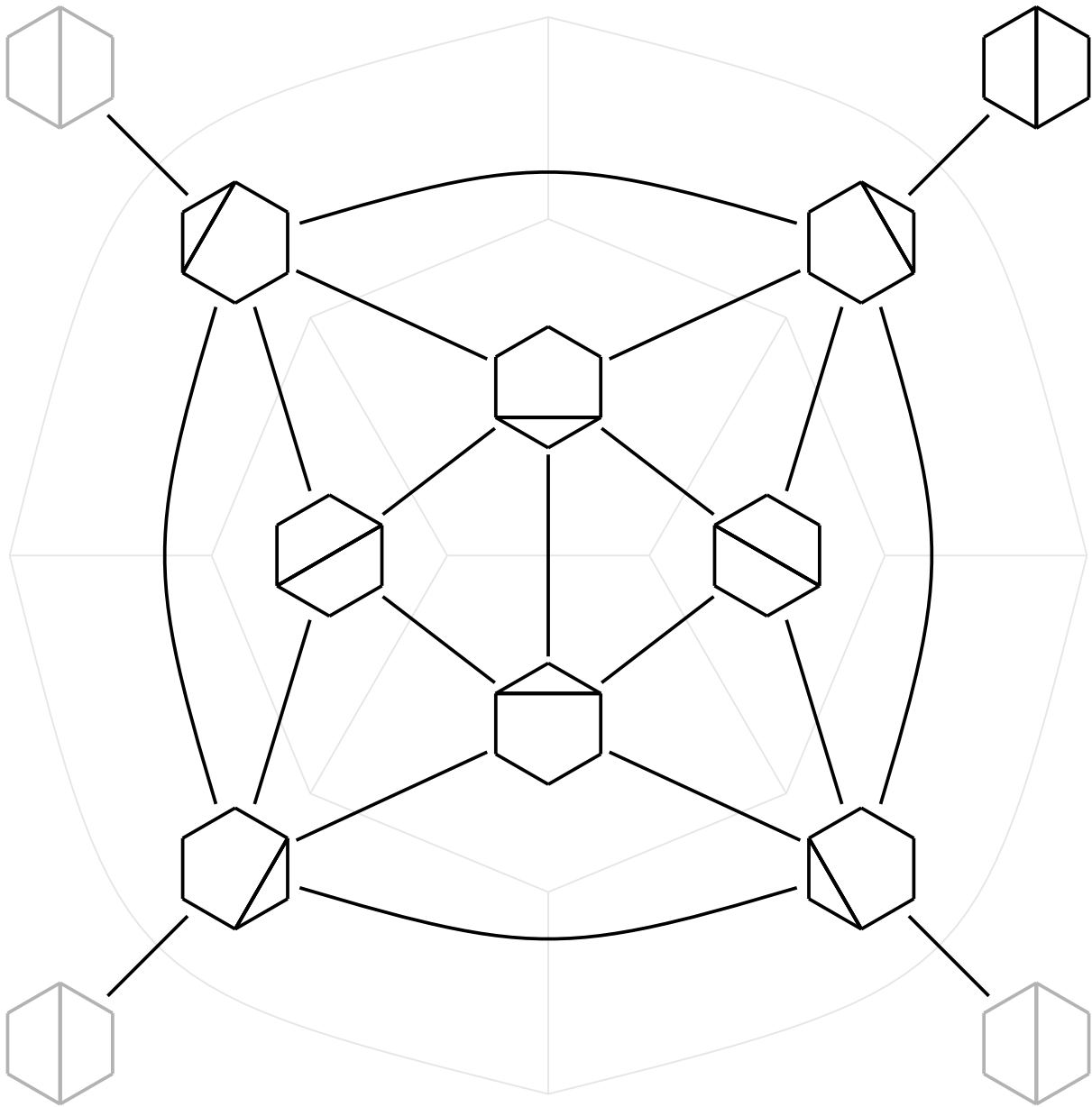
The *arc complex* $\Delta^\circ(\mathbf{S}, \mathbf{M})$ is the *clique complex* for the compatibility relation. This is a *pure simplicial complex*. Its maximal simplices are the ideal triangulations.

$\Delta^\circ(\mathbf{S}, \mathbf{M})$ is a *pseudomanifold with boundary*: each simplex of codimension 1 is contained in at most two maximal simplices.

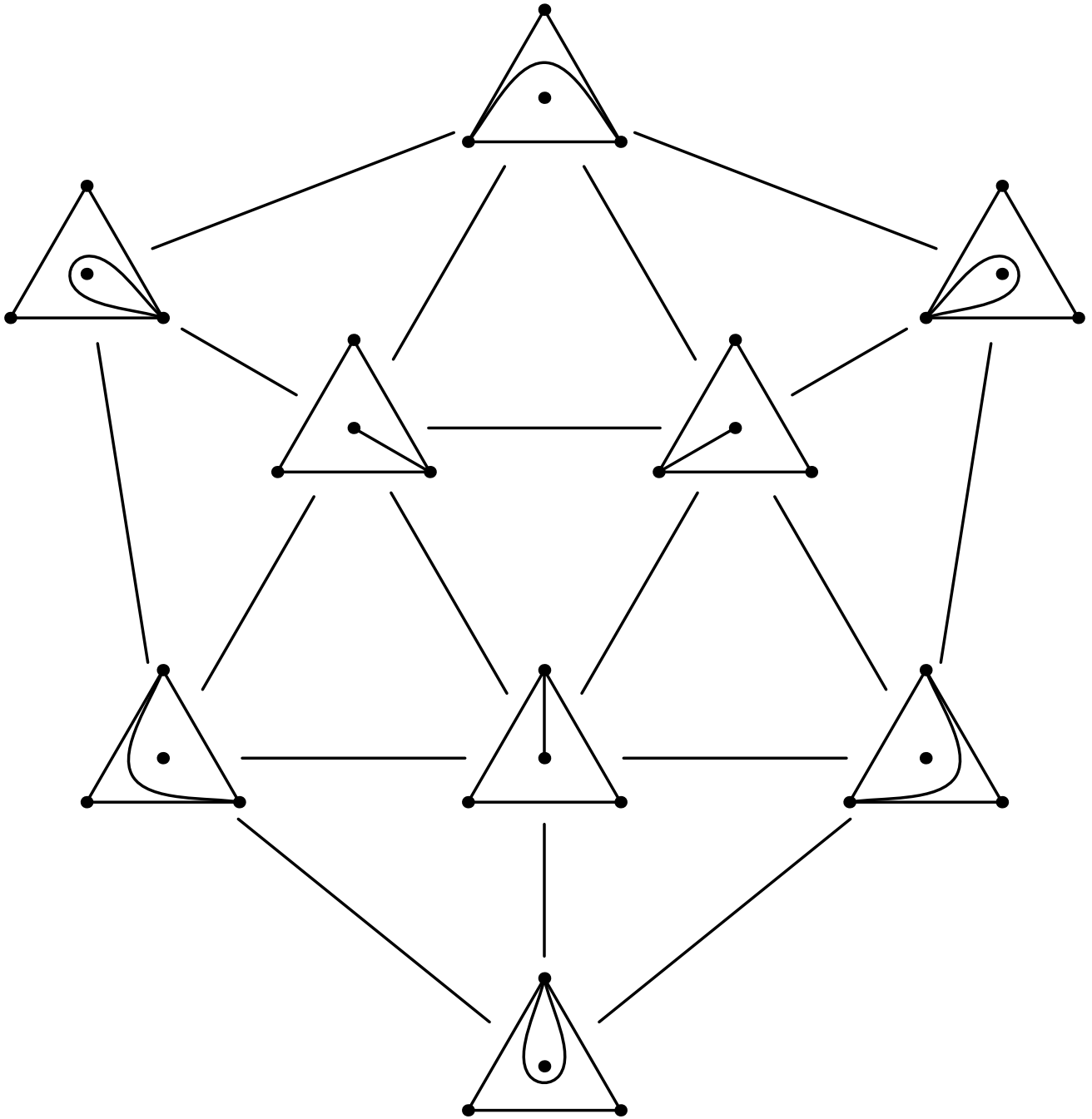
In the special case of an $(n + 3)$ -gon (i.e., an unpunctured disk with $n + 3$ marked points on the boundary), the arc complex is the (polar) dual of an n -dimensional *associahedron*, a.k.a. *Stasheff polytope*. So it is homeomorphic to a sphere. This special case is however exceptional:

Theorem 1 [J.Harer, A.Hatcher] $\Delta^\circ(\mathbf{S}, \mathbf{M})$ is *contractible except when (\mathbf{S}, \mathbf{M}) is a polygon*.

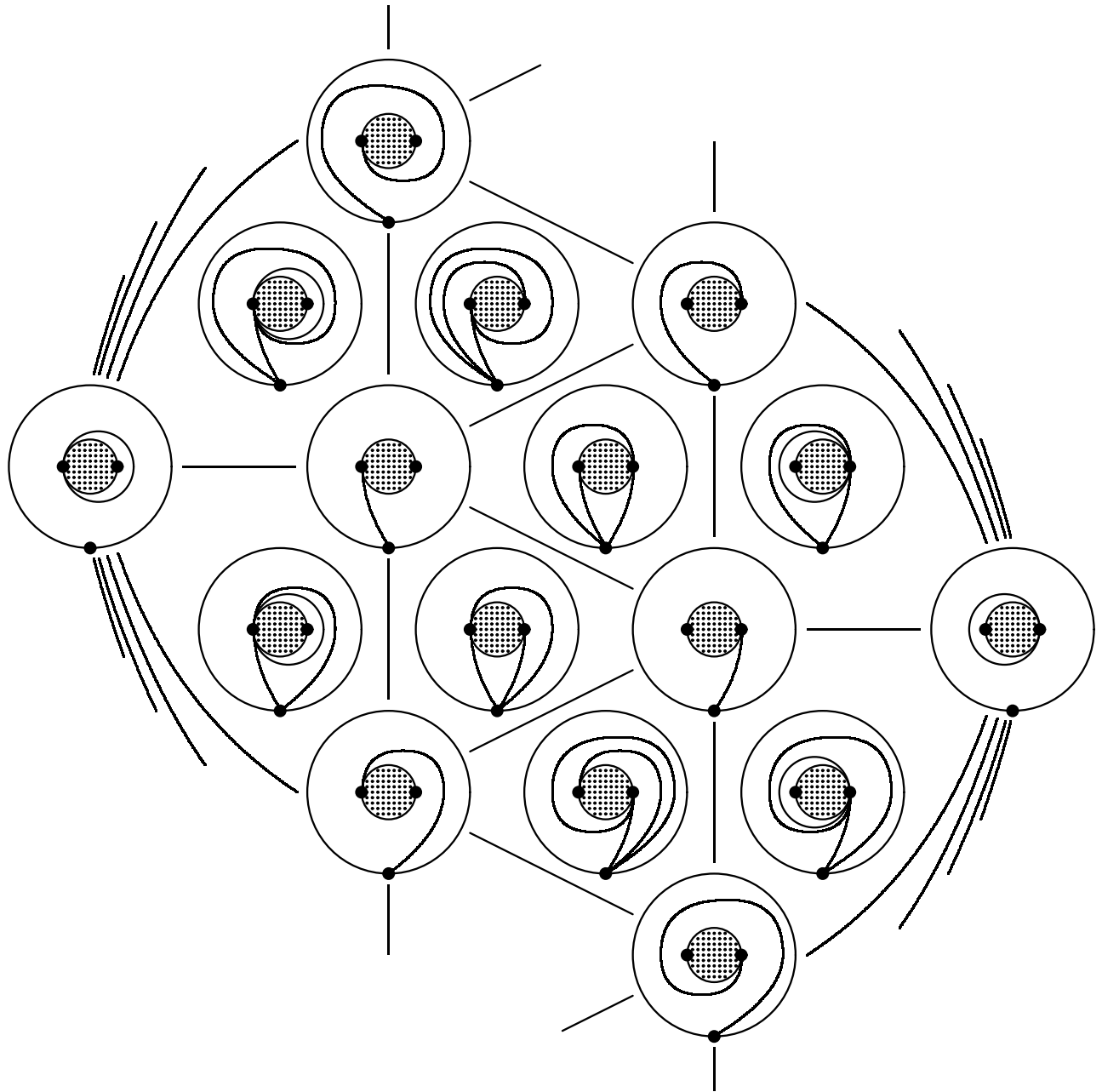
Arc complex of a hexagon



Arc complex of a once-punctured triangle

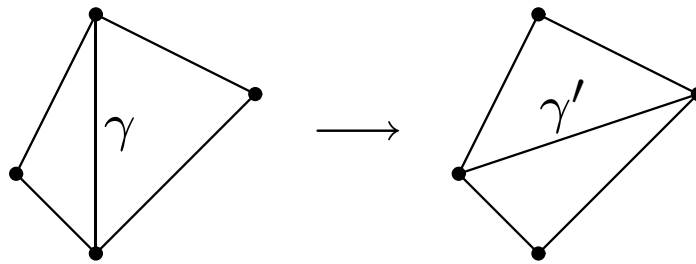


Arc complex for an annulus with $2 + 1$ marked points on the boundary



Dual graph. Flips

The *dual graph* of a pseudomanifold has maximal simplices as vertices, with edges connecting maximal simplices sharing a codimension 1 face. In our case, edges correspond to *flips*.

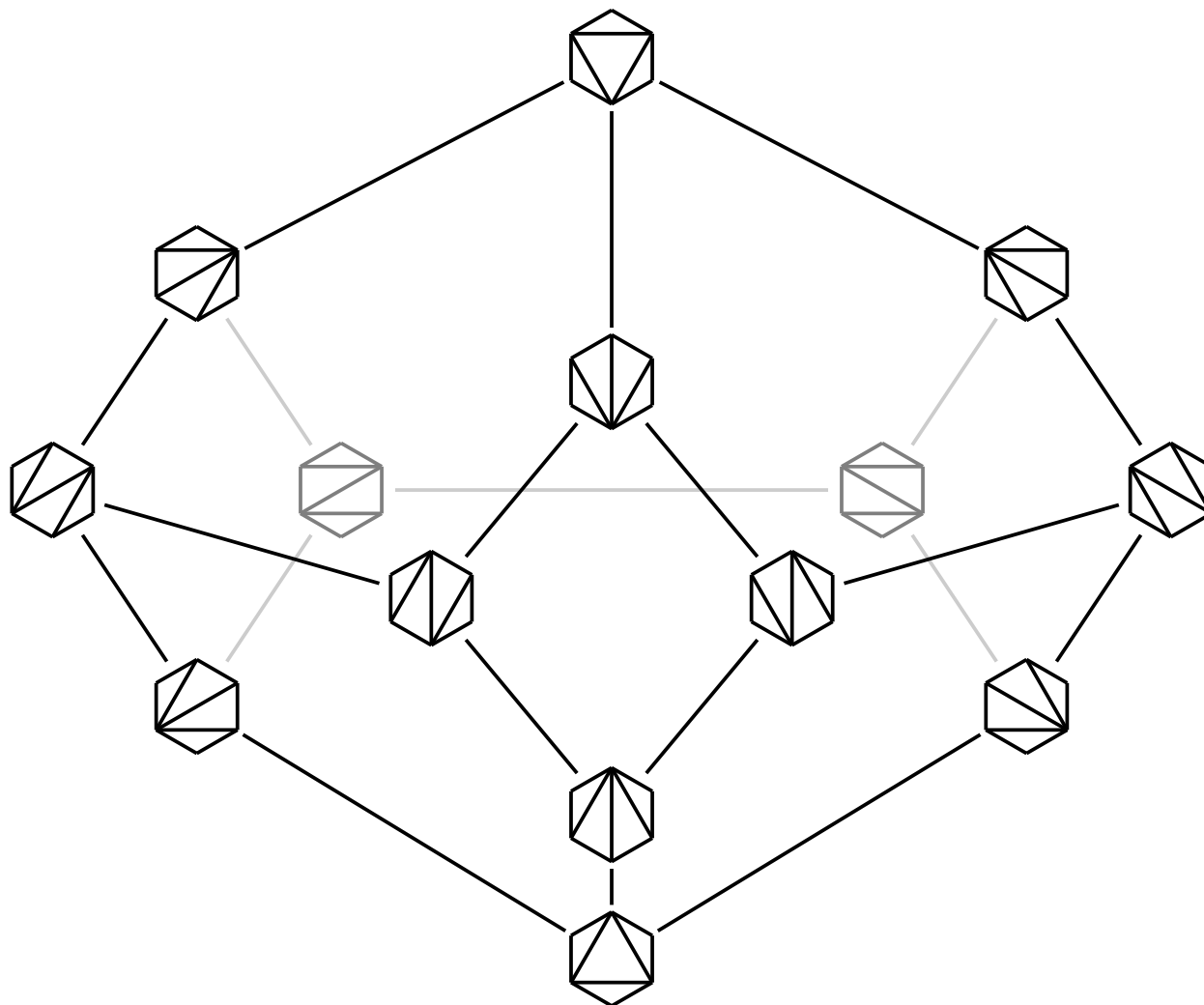


A *flip* is a transformation of an ideal triangulation T that removes an arc γ and replaces it by a unique arc $\gamma' \neq \gamma$ which, together with the remaining arcs, forms a new ideal triangulation T' .

The vertices of the dual graph of $\Delta^\circ(\mathbf{S}, \mathbf{M})$ are labeled by the ideal triangulations of (\mathbf{S}, \mathbf{M}) ; the edges correspond to the flips.

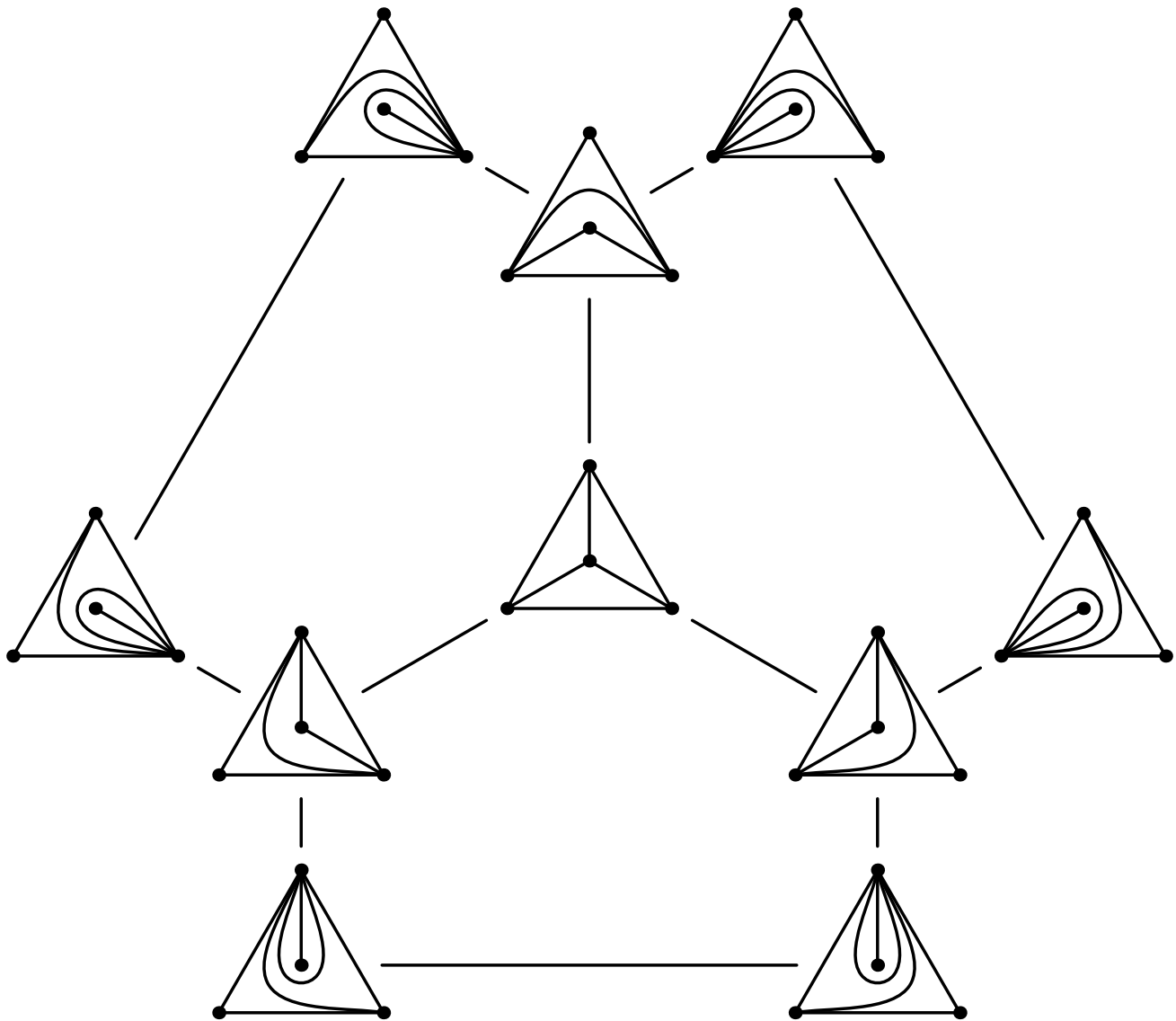
In the case of an unpunctured polygon, we obtain the 1-skeleton of the associahedron.

Dual graph of the arc complex of a hexagon

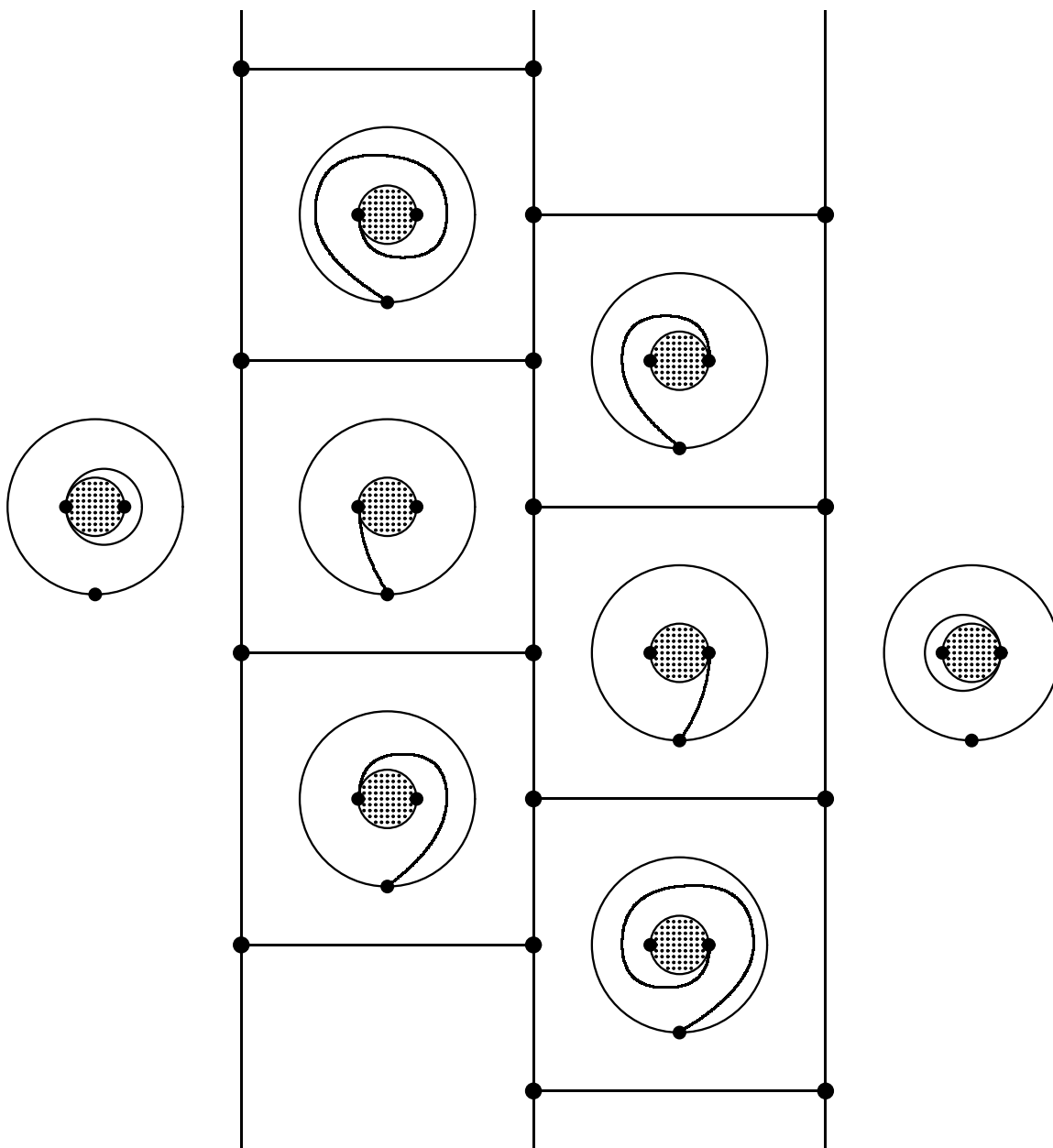


3-dimensional associahedron

Dual graph of the arc complex of a once-punctured triangle



**Dual graph of the arc complex
for an annulus with $2 + 1$ marked points**



2. Cluster complexes

Arc complex vs. cluster complex

Arc complexes play important role in topology of surfaces, Teichmüller theory, the study of moduli spaces, etc.

I would like to advertise another closely related object: the *cluster complex* of a bordered surface with marked points. These complexes arise naturally in the study of *cluster algebras*.

Digression: Cluster algebras

Cluster algebras are a certain class of commutative rings equipped with some additional combinatorial data. After their introduction in

S.Fomin and A.Zelevinsky, Cluster algebras I, *J. Amer. Math. Soc.* **15** (2002), 497–529,

cluster-algebraic structures have been identified and explored in several mathematical disciplines, including:

- Lie theory and quantum groups;
- Quiver representations;
- Poisson geometry and Teichmüller theory;
- Algebraic and geometric combinatorics.

Cluster algebras associated with surfaces

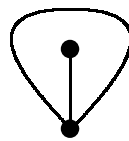
Cluster-algebraic structures associated with triangulated surfaces were first discovered and studied in:

V. V. Fock and A. B. Goncharov, Moduli spaces of local systems and higher Teichmüller theory, *Publ. Math. IHES* **103** (2006), 1–211;

M. Gekhtman, M. Shapiro, and A. Vainshtein, Cluster algebras and Weil-Petersson forms, *Duke Math. J.* **127** (2005), 291–311.

Beyond the arc complex

The arc complex $\Delta^\circ(\mathbf{S}, \mathbf{M})$ describes the combinatorics of triangulations and flips. However, not every arc can be flipped: we cannot flip the “folded” side of a self-folded ideal triangle.



This accounts for the boundary of $\Delta^\circ(\mathbf{S}, \mathbf{M})$.

Let us try to extend the arc complex beyond its boundary, by introducing some “imaginary arcs” obtained by flips that are not normally allowed. We will then need to introduce further such objects obtained by subsequent flips, etc.

In order to understand what those objects are (or even define them properly), we will recast the whole picture in formal algebraic terms.

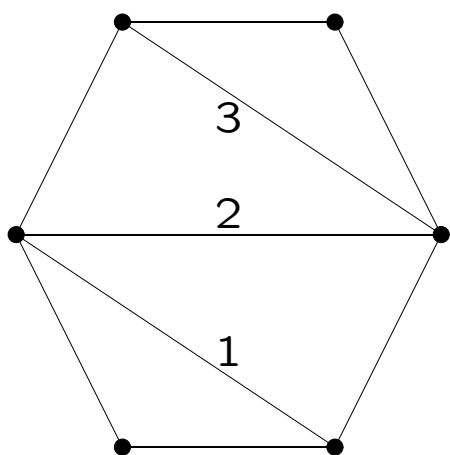
Signed adjacency matrix

The *signed adjacency matrix* $B(T)$ of an ideal triangulation T is a certain $n \times n$ skew-symmetric integer matrix whose rows and columns correspond to the arcs in T . (It will be convenient to label these arcs by the numbers $1, \dots, n$.)

We first define $B(T)$ under the following simplifying assumptions:

- no self-folded triangles;
- no two triangles in T share two sides.

The entries of $B(T) = (b_{ij})$ describe the *signed adjacencies* between the corresponding arcs.



$$B(T) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Definition of $B(T)$, general case

We set

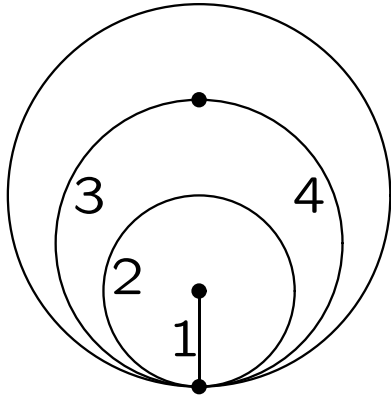
$$B(T) \stackrel{\text{def}}{=} \sum B^\Delta,$$

the sum over all non-self-folded ideal triangles Δ in T ; the matrices B^Δ are defined as follows.

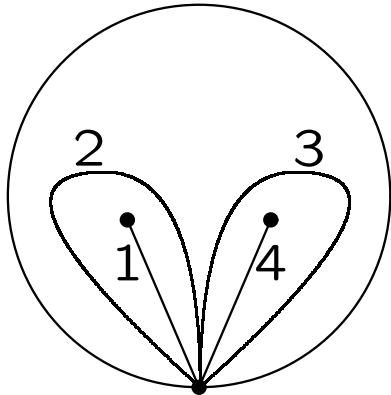
For an arc i in T , let $\pi_T(i)$ be the following arc: if there is a triangle in T folded along i , then $\pi_T(i)$ is its enclosing loop; otherwise, $\pi_T(i) = i$. Now, $B^\Delta = (b_{ij}^\Delta)$ is defined by

$$b_{ij}^\Delta = \begin{cases} 1 & \text{if } \Delta \text{ has sides } \pi_T(i) \text{ and } \pi_T(j), \\ & \text{with } \pi_T(j) \text{ following } \pi_T(i) \\ & \text{in the clockwise order;} \\ -1 & \text{same, counter-clockwise;} \\ 0 & \text{otherwise.} \end{cases}$$

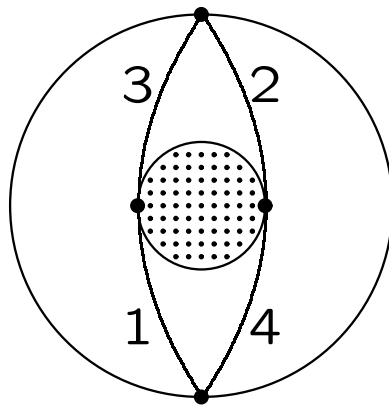
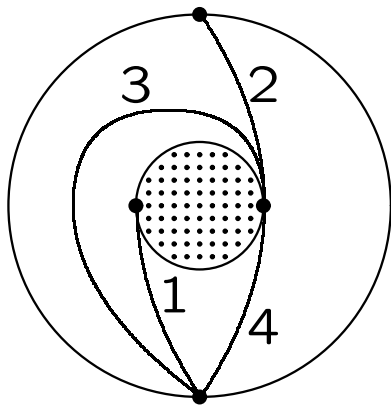
Matrices $B(T)$: examples



$$B(T) = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$



$$B(T) = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$



Mutations and flips

Now that we replaced triangulations by matrices, we need to say what transformations of these matrices correspond to flips.

We say that an $n \times n$ matrix $B' = (b'_{ij})$ is obtained from a matrix B by *matrix mutation* in direction k , and write $B' = \mu_k(B)$, if

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Proposition 2 *If an ideal triangulation T' is obtained from T by a flip that replaces an arc labeled k by a new arc, then $B(T') = \mu_k(B(T))$.*

This suggests how to perform a forbidden flip: just mutate $B(T)$ in the corresponding direction.

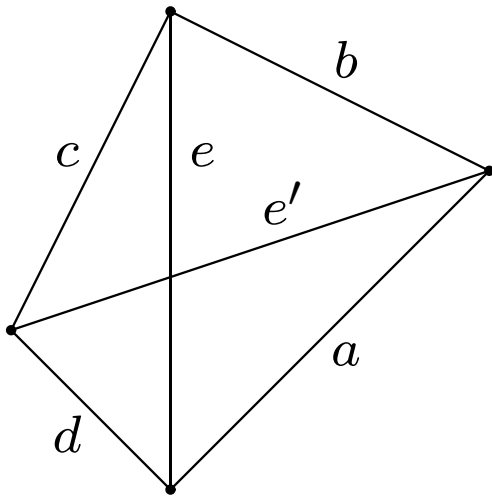
Cluster variables

Question: How shall we recognize if we have come back to the same arc (be it ordinary or “imaginary”) as a result of consecutive flips?

Answer: By assigning to each arc a a rational function x_a , called a *cluster variable*, which will serve as a “certificate of identity” for γ .

Generalized Ptolemy relations

Cluster variables will satisfy *Ptolemy relations*:



$$x_e x_{e'} = x_a x_c + x_b x_d$$

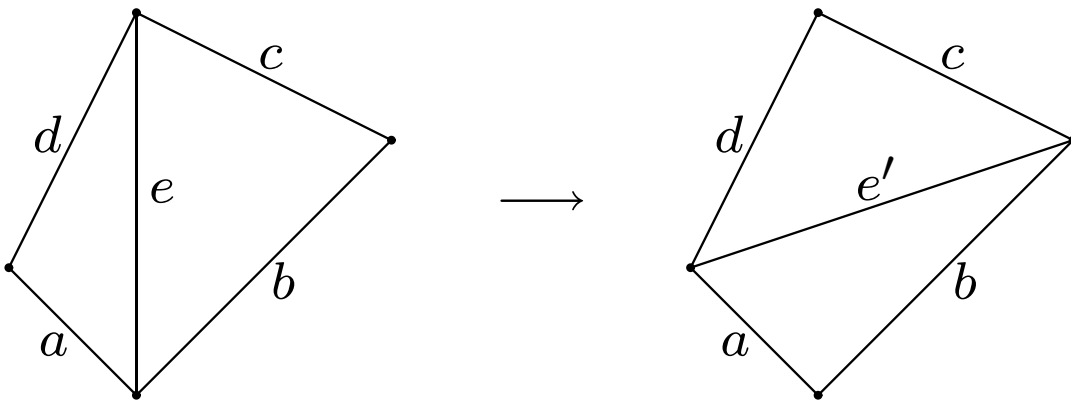
(If p is a boundary segment, set $x_p = 1$.)

In general:

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}.$$

Defining cluster variables

For triangulations related by a flip, the corresponding n -element sets of cluster variables (called *clusters*) can be expressed birationally in terms of each other:



$$x_{e'} = \frac{x_a x_c + x_b x_d}{x_e}.$$

Lemma 3 *Birational maps associated to all possible flips form a commuting diagram.*

Consequently, the cluster variables assigned to the ordinary arcs are well defined (as elements of a field of rational functions in n variables).

Digression: Lambda-lengths

[R.Penner, V.V.Fock]

There is a geometric reason why the cluster variables are well defined. They can actually be given an intrinsic geometric interpretation, not to be discussed in detail in this talk.

Define a hyperbolic structure on the surface so that all boundary components are made of geodesics connecting cusps at marked points. The distance along a geodesic between two marked points is infinite, so we renormalize by picking an *horocycle* around each cusp and measuring the distance between horocycles. We then obtain the *lambda lengths* of the geodesics. These lambda lengths turn out to satisfy the Ptolemy relations, hence can be identified with the cluster variables. They are the *Penner coordinates* on the corresponding *decorated Teichmüller space*.

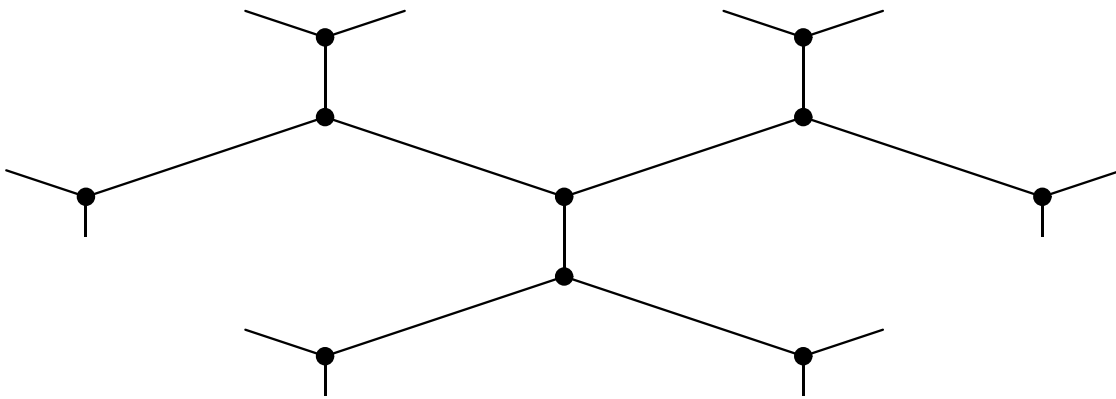
Seeds

We can now achieve our goal of allowing flips in all directions. Identify each triangulation T with a pair (B, \mathbf{x}) , called a *seed*, consisting of the signed adjacency matrix $B = B(T)$ and the corresponding cluster $\mathbf{x} = \mathbf{x}(T) = \{x_1, \dots, x_n\}$. Flipping an arc corresponding to x_k amounts to the following *seed mutation*:

- compute the new matrix $B' = \mu_k(B)$;
- compute the new cluster $\mathbf{x}' = \mathbf{x} - \{x_k\} \cup \{x'_k\}$ using the corresponding Ptolemy relation.

Cluster algebra and cluster complex

Iterating this process indefinitely in all possible directions, we obtain a collection of seeds (B, \mathbf{x}) that play the role of “generalized triangulations” of our surface. The elements of all clusters \mathbf{x} appearing in all of these seeds are still called cluster variables. The ring they generate is an example of a *cluster algebra*.



Theorem 4 *Each seed (B, \mathbf{x}) is uniquely determined by its cluster \mathbf{x} .*

Define the *cluster complex* $\Delta(\mathcal{S}, \mathcal{M})$ as the simplicial complex on the set of all cluster variables whose maximal simplices are the clusters.

Digression: finite type classification [SF-Zelevinsky]

The iterative construction of seed mutations can be applied without any modifications to an initial seed (B, \mathbf{x}) in which B is any skew-symmetrizable matrix, not necessarily coming from a surface.

All instances in which the set of seeds is finite (so the corresponding cluster algebra is of *finite type*) can be classified. This classification turns out to be completely parallel to the Cartan-Killing classification.

Among finite types, only types A_n and D_n (in the Cartan-Killing nomenclature) arise within the framework of triangulated surfaces.

Theorem 5 $\Delta(\mathbf{S}, \mathbf{M})$ is finite if and only if (\mathbf{S}, \mathbf{M}) is a disk or a once-punctured disk.

In these two cases, we recover, respectively, the associahedra of types A_n and D_n .

3. Main results

Theorem 6 *The cluster complex $\Delta(\mathbf{S}, \mathbf{M})$ is a pseudomanifold. Its dual graph is connected.*

Theorem 7 *$\Delta(\mathbf{S}, \mathbf{M})$ is a flag complex, i.e., it is the clique complex for its 1-skeleton.*

Theorem 8 *$\Delta(\mathbf{S}, \mathbf{M})$ is either contractible or homotopy equivalent to a sphere.*

Next:

- a concrete description of $\Delta(\mathbf{S}, \mathbf{M})$ in terms of combinatorial topology of (\mathbf{S}, \mathbf{M}) ;
- a more specific version of Theorem 8.

Tagged arcs

Each arc γ in (S, M) has two *ends*. A *tagged arc* is an arc in which each end has been tagged in one of two ways, *plain* or *notched*, so that the following conditions are satisfied:

- the arc does not cut out a once-punctured monogon;
- an endpoint lying on the boundary must be tagged plain;
- both ends of a loop must be tagged in the same way.

Tagged arc complex

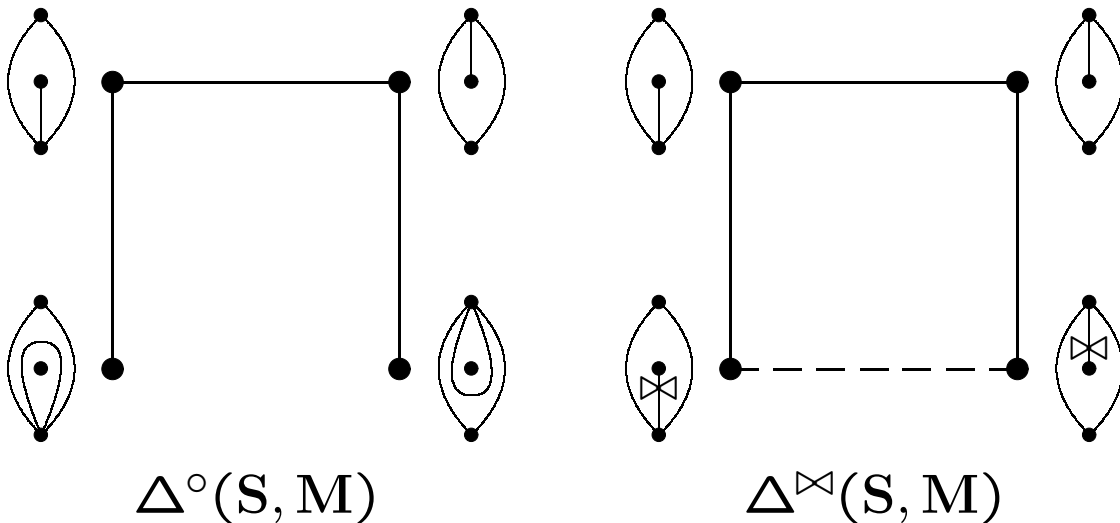
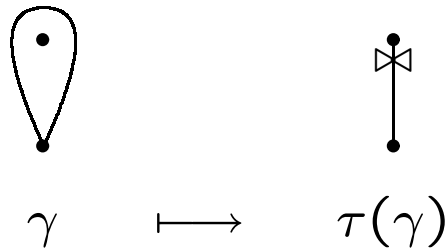
Tagged arcs α and β are *compatible* iff:

- the untagged versions of α and β are compatible;
- if the untagged versions of α and β are different, and α and β share an endpoint a , then the ends of α and β connecting to a must be tagged in the same way;
- if the untagged versions of α and β coincide, then at least one end of α must be tagged in the same way as the corresponding end of β .

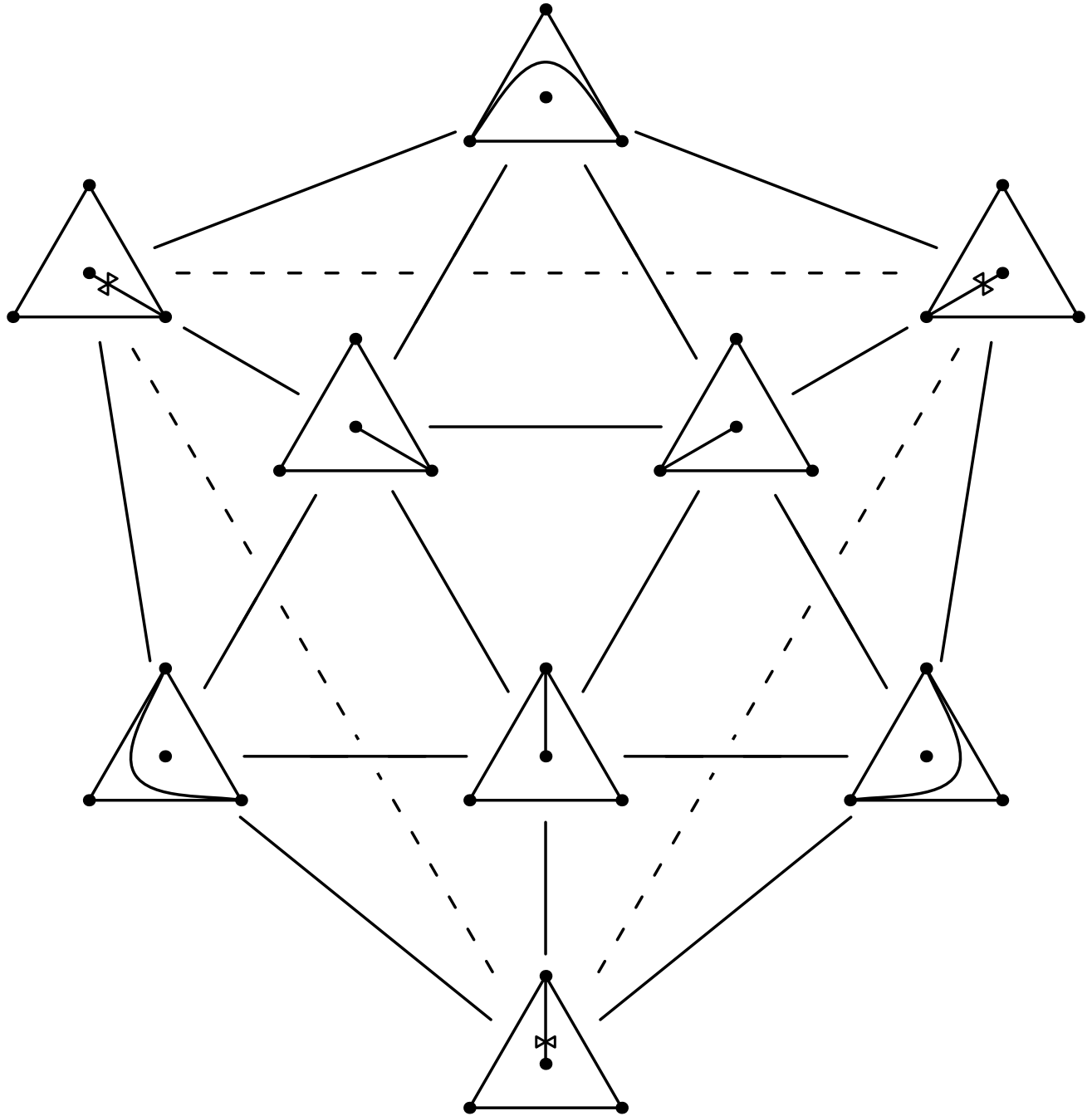
The *tagged arc complex* $\Delta^{\bowtie}(\mathbf{S}, \mathbf{M})$ is the clique complex for the compatibility relation on the set of tagged arcs. Its maximal simplices are called *tagged triangulations*.

Ordinary vs. tagged arc complex

$\Delta^\circ(\mathbf{S}, \mathbf{M})$ can be embedded into $\Delta^{\boxtimes}(\mathbf{S}, \mathbf{M})$ (as a *non-induced* subcomplex) via the following map $\gamma \mapsto \tau(\gamma)$ from ordinary arcs to tagged arcs. If γ does not cut out a once-punctured monogon, then $\tau(\gamma) = \gamma$ (both tags plain). Otherwise:



Tagged arc complex of a once-punctured triangle



$\Delta^{\bowtie}(\mathbf{S}, \mathbf{M})$ as the cluster complex

Theorem 9 *Assume that (\mathbf{S}, \mathbf{M}) is not a closed surface with one puncture. Then the cluster complex $\Delta(\mathbf{S}, \mathbf{M})$ is canonically isomorphic to the tagged arc complex $\Delta^{\bowtie}(\mathbf{S}, \mathbf{M})$.*

Thus, the cluster variables in the corresponding cluster algebra can be labeled by the tagged arcs in (\mathbf{S}, \mathbf{M}) , while the clusters correspond to the tagged triangulations of (\mathbf{S}, \mathbf{M}) .

In the case of a closed surface with exactly one puncture, $\Delta^{\bowtie}(\mathbf{S}, \mathbf{M})$ consists of 2 connected components, each isomorphic to $\Delta(\mathbf{S}, \mathbf{M})$.

Topology of the cluster complex

Here is a more concrete version of Theorem 8.

Theorem 10 *The tagged arc complex $\Delta^{\bowtie}(\mathbf{S}, \mathbf{M})$ is either contractible or homotopy equivalent to a sphere. Specifically:*

- *If (\mathbf{S}, \mathbf{M}) is a polygon or a once-punctured polygon, then $\Delta^{\bowtie}(\mathbf{S}, \mathbf{M})$ is homeomorphic to an $(n-1)$ -dimensional sphere S^{n-1} .*
- *If (\mathbf{S}, \mathbf{M}) is a closed surface with p punctures, then $\Delta^{\bowtie}(\mathbf{S}, \mathbf{M})$ is homotopy equivalent to S^{p-1} .*
- *In all other cases, $\Delta^{\bowtie}(\mathbf{S}, \mathbf{M})$ is contractible.*

Growth rate of the cluster complex

A cluster complex has *polynomial growth* if the number of distinct seeds which can be obtained from a fixed initial seed by at most n mutations is bounded from above by a polynomial in n . A cluster complex has *exponential growth* if the number of such seeds is bounded from below by an exponentially growing function of n .

A *feature* of a surface is either a puncture or a boundary component.

Theorem 11 *The cluster complex has polynomial growth for spheres with at most 3 features. In all other cases, it has exponential growth.*

To restate, the cases of polynomial growth are: a disk with ≤ 2 punctures, an annulus with ≤ 1 puncture, and a pair of pants.

Proof techniques

- Direct combinatorial arguments
- The Nerve Lemma [Borsuk, Björner]
- Contractibility of the arc complex [Harer, Hatcher]
- Fundamental group of the graph of flips [Chekhov-Penner, Harer]
- Generalized Ptolemy relations [Penner]
- Iterated “combinatorial Mayer-Vietoris”
- Tits alternative for mapping class groups [Ivanov, McCarthy]
- The Laurent phenomenon [SF-Zelevinsky]
- Interpreting denominator exponents as generalized intersection numbers

Generalized cluster complexes

In finite type, there is a natural generalization of cluster complexes introduced and studied by SF-N.Reading and C.Athanasiadis-E.Tzanaki. (Subsequent work by C.Krattenthaler, B.Zhu, K.Baur-R.Marsh, H.Thomas *et al.*)

For example, in type A_n one replaces triangulations of a polygon by partitions into $(m+2)$ -gons, where m is a positive integer parameter.

The topology of these generalized cluster complexes is well understood: they are homotopy equivalent to wedges of spheres.

It would be interesting to obtain analogues of these results for general bordered surfaces.