The Schur expansion for Macdonald polynomials

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The *transformed Macdonald polynomials* $\widetilde{H}_\mu(x;q,t)$ are the unique functions satisfying the following conditions:

(i) $\widetilde{H}_\mu(x;q,t) \in \mathbb{Q}(q,t)\{s_\lambda[X/(1-q)] \;:\; \lambda \geq \mu\}$,

(ii) $\widetilde{H}_\mu(x;q,t) \in \mathbb{Q}(q,t)\{s_\lambda[X/(1-t)] \;:\; \lambda \geq \mu'\}$,

(iii) $\widetilde{H}_\mu[1;q,t] = 1$.

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The *Kostka-Macdonald polynomials* $\widetilde{K}_{\lambda,\mu}(q,t)$ give the Schur expansion for Macdonald polynomials, i.e.

$$\widetilde{H}_\mu(x;q,t) \ = \ \sum_\lambda \widetilde{K}_{\lambda,\mu}(q,t) s_\lambda(x).$$

Theorem. (Haiman 2001)

$$\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$$

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Better yet, find a combinatorial formula for $\widetilde{K}_{\lambda,\mu}(q,t)$.

For S a filling of the Young diagram of μ, define

$$\begin{aligned} \operatorname{maj}(S) &\overset{\text{def}}{=} |\operatorname{Des}(S)| + \sum_{c \in \operatorname{Des}(S)} l(c), \\ \operatorname{inv}(S) &\overset{\text{def}}{=} |\operatorname{Inv}(S)| - \sum_{c \in \operatorname{Des}(S)} a(c). \end{aligned}$$

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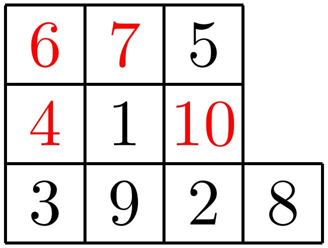

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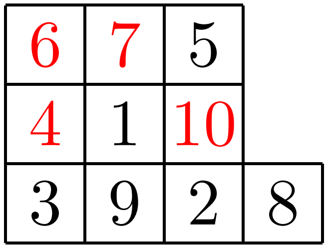

$$\begin{array}{|c|c|c|c} \hline {\color{red}6} & {\color{red}7} & 5 & \\ \hline {\color{red}4} & 1 & {\color{red}10} & \\ \hline 3 & 9 & 2 & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

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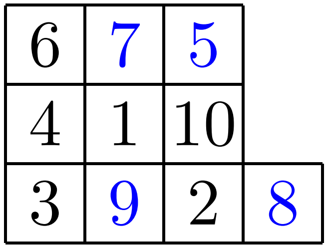

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inversions

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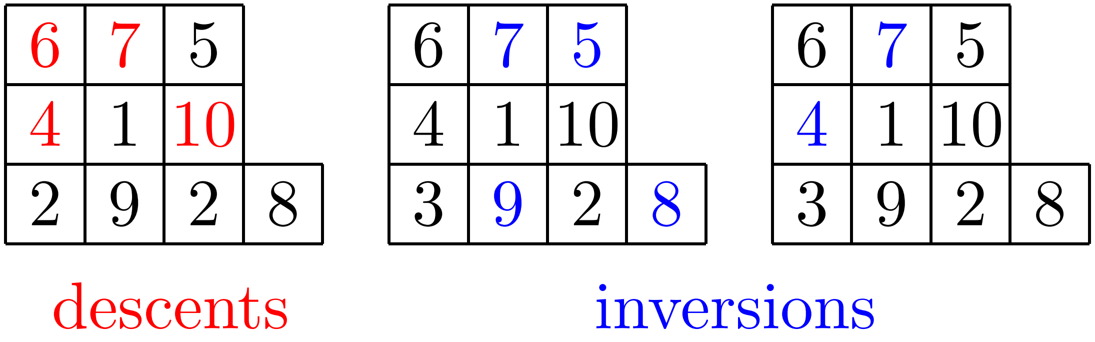

Theorem. (Haglund, Haiman, Loehr 2005)

$$\widetilde{H}_\mu(x;q,t) \quad = \quad \sum_{S:\mu\to\mathbb{N}} q^{\mathrm{inv}(S)} t^{\mathrm{maj}(S)} x^S$$

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The Schur functions may be defined by

$$s_\lambda(x) \quad = \quad \sum_{T\in\mathrm{SSYT}(\lambda)} x^T$$

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Proposition. (Gessel 1984)

$$\begin{aligned} s_\lambda(x) \quad &= \quad \sum_{T\in\mathrm{SSYT}(\lambda)} x^T \\ &= \quad \sum_{T\in\mathrm{SYT}(\lambda)} Q_{n,D(T)}(x) \end{aligned}$$

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For $D \subset \{1, 2, \ldots, n-1\}$, Gessel defined the *quasi-symmetric function* $Q_{n,D}(x)$ by

$$Q_{n,D}(x) \ = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ i_j = i_{j+1} \Rightarrow j \notin D}} x_{i_1} \cdots x_{i_n}.$$

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Define the *descent signature* $\sigma \; : \; \mathrm{SYT} \rightarrow \{\pm 1\}^{n-1}$ by

$$\sigma(T)_i \; = \begin{cases} +1 & i \text{ left of } i{+}1 \text{ in } w(T) \\ -1 & i{+}1 \text{ left of } i \text{ in } w(T) \end{cases}$$

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The *descent set* of T is $D(T) = \{i \mid \sigma(T)_i = -1\}$.

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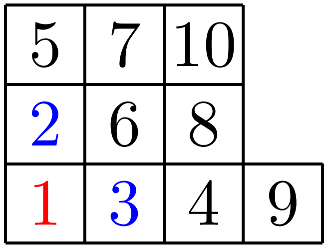

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$$\begin{array}{|c|c|c|c}
\hline
5 & 7 & 10 & \multicolumn{1}{|c}{} \\
\hline
\color{blue}{2} & 6 & 8 & \multicolumn{1}{|c}{} \\
\hline
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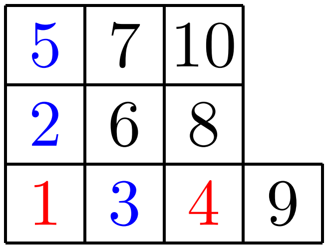

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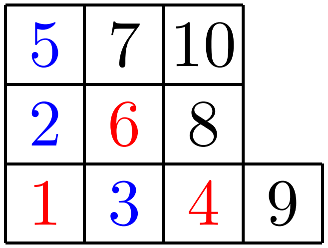

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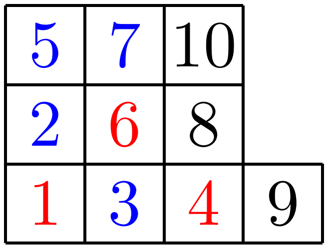

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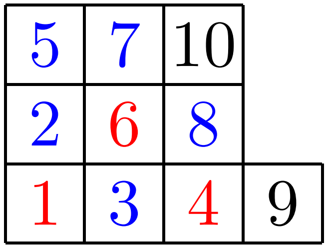

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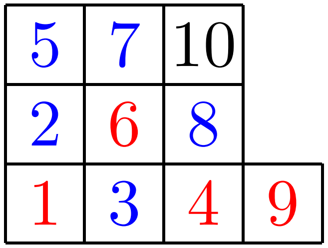

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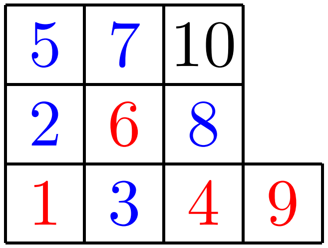

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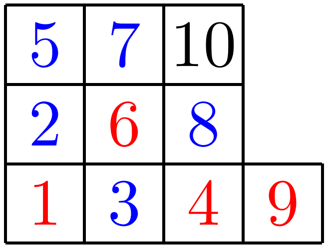

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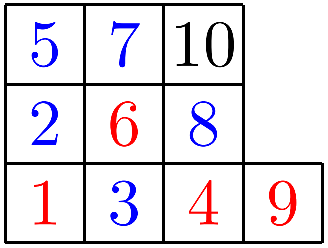

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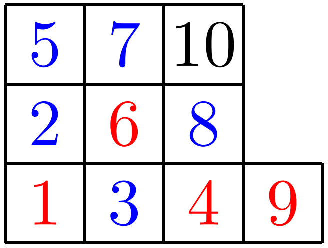

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Define the *generating function* of $\mathcal{G}$ by

$$g(x) \ = \ \sum_{v \in V(\mathcal{G})} Q_{n,\sigma(v)}(x).$$

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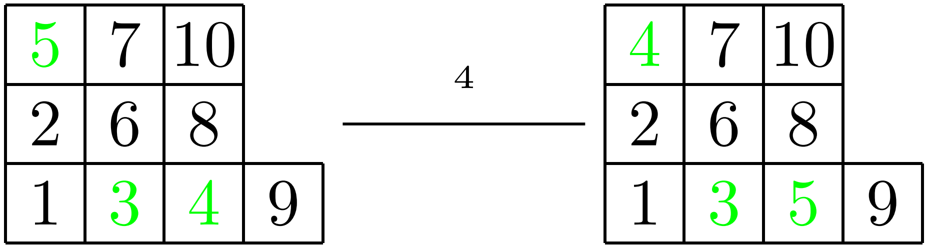

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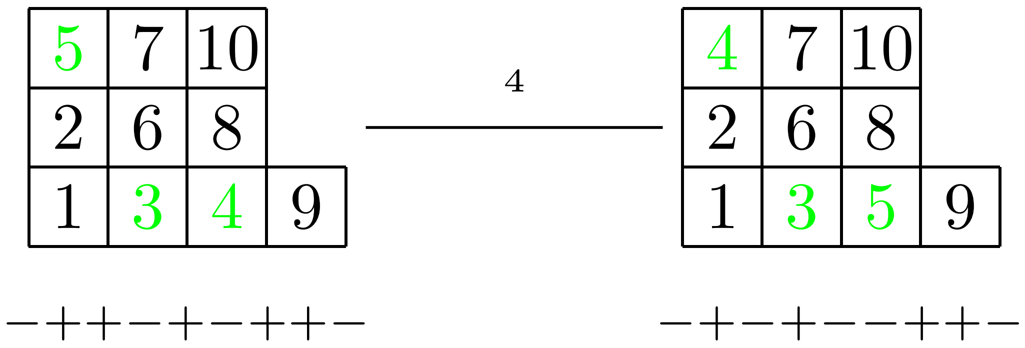

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Define a vertex-signed, edge-colored graph $\mathcal{G} = (V, \sigma, E)$ whose connected components are given by $\mathcal{G}_\lambda$.

Definition. A vertex-signed, edge-colored graph $\mathcal{G}$ is a *dual equivalence graph* if it satisfies 5 local axioms about signatures and edge colors.

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Theorem. (A.) Every connected component of a DEG is isomorphic to $\mathcal{G}_\lambda$ for a unique partition λ.

Definition. A vertex-signed, edge-colored graph $\mathcal{G}$ is a *dual equivalence graph* if it satisfies 5 local axioms about signatures and edge colors.

Theorem. (A.) Every connected component of a DEG is isomorphic to $\mathcal{G}_\lambda$ for a unique partition λ.

Corollary. (A.) If $\mathcal{G}$ is a DEG and α, β are statistics on $V(\mathcal{G})$ which are constant on connected components, then

$$\sum_{v \in V(\mathcal{G})} q^{\alpha(v)} t^{\beta(v)} Q_{n,\sigma(v)}(x) = \sum_{\lambda} \left(\sum_{\mathcal{C} \cong \mathcal{G}_\lambda} q^{\alpha(\mathcal{C})} t^{\beta(\mathcal{C})} \right) s_{\lambda(x)}.$$

Recall Haglund's formula

$$\widetilde{H}_{\mu}(x;q,t) \;=\; \sum_{S:\mu \tilde{\to} [n]} q^{\mathrm{inv}(S)} t^{\mathrm{maj}(S)} Q_{n,\sigma(S)}(x).$$

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$\mathcal{H}_\mu = (V, \sigma, E)$ must be a DEG

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$$r(S) \;=\; 5 \;\; 7 \;\; 10 \;\; 0 \;\; 2 \;\; 6 \;\; 8 \;\; 0 \;\; 1 \;\; 3 \;\; 4 \;\; 9$$

Define involutions

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$$
\begin{array}{cccccccc}
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\end{array}
$$

$$
D_i^{(k)}(w) \;=\; \begin{cases} d_i(w) & \text{if } \operatorname{dist}(i-1,i,i+1) > k \\ \widetilde{d}_i(w) & \text{if } \operatorname{dist}(i-1,i,i+1) \le k \end{cases}
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$$\begin{aligned} |\mathrm{Inv}(S)| &= \left|\mathrm{Inv}\left(D_i^{(\mu_1)}(S)\right)\right| \\ \mathrm{Des}(S) &= \mathrm{Des}\left(D_i^{(\mu_1)}(S)\right) \end{aligned}$$

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$$\begin{array}{c} |\mathrm{Inv}(S)| = \left|\mathrm{Inv}\left(D_i^{(\mu_1)}(S)\right)\right| \\ \mathrm{Des}(S) = \mathrm{Des}\left(D_i^{(\mu_1)}(S)\right) \end{array} \quad \rightsquigarrow \quad \begin{array}{c} \mathrm{inv}(S) = \mathrm{inv}\left(D_i^{(\mu_1)}(S)\right) \\ \mathrm{maj}(S) = \mathrm{maj}\left(D_i^{(\mu_1)}(S)\right) \end{array}$$

Almost define i-colored edges to be the pairs

$$\left\{ S, D_i^{(\mu_1)}(S) \right\},$$

but with a bit of tweaking when $\mu_1 \geq 3$.

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Conjecture. $\mathcal{H}_\mu$ is a DEG for which inv and maj are constant on connected components.

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Corollary. (A.) For $\mu_1 \leq 3$, we have

$$\widetilde{K}_{\lambda,\mu}(q,t) \; = \; \sum_{\mathcal{C} \cong \mathcal{G}_\lambda} q^{\mathrm{inv}(\mathcal{C})} t^{\mathrm{maj}(\mathcal{C})}.$$

THAT'S ALL FOLKS!