The divergence theorem

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Reading for this week

- Sections 8.2, 8.3, 8.4
- Common final: Thursday, May 7 from 7:10–9:00 p.m., PSA 109

In case of the flu (asu.edu/pandemic)

- If you don't feel well, don't come to class!
- If you think you have the flu on final exam day, please see a doctor and get a note
- Please notify me by email or telephone! Otherwise you'll get a 0
- A grade of Incomplete will be assigned
- I will arrange a make-up date
- If the university is closed, then I will post more information on the course web site

Important theorem about irrotational vector fields

- Suppose **F** is a smooth vector field with at most a finite number of exceptional points. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0 \iff \text{path independence}$$
$$\iff \mathbf{F} = \nabla f$$
$$\iff \nabla \times \mathbf{F} = \mathbf{0}$$

- Key example: $\mathbf{F} = \nabla \left(\dfrac{\text{constant}}{\|\mathbf{r}\|} \right)$

Discussion questions

Determine whether each vector field is irrotational, and if so, then find a potential function.

1. $\mathbf{F} = -y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$
2. $\mathbf{F} = (x^2, y^2, z^2)$
3. $\mathbf{F} = \left(3x^2 \sin(xy) + x^3 y \cos(xy)\right)\mathbf{i} + x^4 \cos(xy)\mathbf{j}$
4. $\mathbf{F} = \dfrac{(x, y)}{\sqrt{x^2 + y^2}}$

The divergence

- Let $\mathbf{F} = (P, Q, R)$ be a smooth vector field
- Then
$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- Example: $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$

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- Then $\operatorname{div} \mathbf{F} = x^2 + y^2$

The divergence theorem (Gauss's theorem)

- Suppose $\mathbf{F}$ is a smooth vector field
- Suppose S is a union of symmetric elementary regions enclosing a volume V
- Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \operatorname{div} \mathbf{F} \, dV$$

Physical interpretation: flux per unit volume

- If $\operatorname{div} \mathbf{F} > 0$ in S, then there is a net outward flux from S (i.e., a source)
- If $\operatorname{div} \mathbf{F} < 0$ in S, then there is a net inward flux (i.e., a sink)
- If $\operatorname{div} \mathbf{F} = 0$ in S, then there is no net flux from S (i.e., a divergence free)
- A divergence-free fluid flow is incompressible (e.g., water flowing through a pipe or channel)

Example

- $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$
- Find the flux of $\mathbf{F}$ out of the unit sphere S
- Use Gauss's theorem:

$$\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iiint_S \operatorname{div} \mathbf{F} \, dV = \iiint_S (x^2 + y^2) \, dV$$

- Switch to spherical coordinates:

$$(x^2 + y^2) \, dV = (1 - \cos^2 \phi)(\rho^2 \sin \phi) \, d\phi \, d\theta \, d\rho$$

Example, continued

- Hence the net flux is

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi} (1 - \cos^2\phi)(\rho^2 \sin\phi)\, d\phi\, d\theta\, d\rho = \frac{8\pi}{9}$$

Discussion questions

1. (#1) Compute the flux of $\mathbf{F} = (x-y, y-z, z-x)$ out of the unit sphere
2. (#3) Compute the flux of $\mathbf{F} = \mathbf{r}$ out of the unit cube (in the first octant) both directly and using Gauss's theorem