Circulation and gradient fields

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Reading for this week

- Sections 8.2, 8.3, 8.4
- Common final: Thursday, May 7 from 7:10–9:00 p.m., PSA 109

Remarks on yesterday's lab

- Suppose S is a smooth, orientable surface with boundary ∂S
- **Stokes' theorem:** If ∂S is positively oriented,

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

- **Note:** If S_1 and S_2 are two different orientable surfaces with the same boundary curve, then

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

  Use the simplest S!

Remarks on “lazy” evaluation of integrals

- Line integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

  does not require $\mathbf{r}'(t)$ to be a unit vector

- The equivalent formulation

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{T}\, ds$$

  requires $\mathbf{T}$ to be a unit tangent vector

- This is because

$$\int_a^b \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\, ds = \int_a^b \mathbf{F} \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\, \|\mathbf{r}'(t)\|\, dt$$

Remarks on “lazy” evaluation of integrals, 2

- Surface integrals:

  $$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot(\mathbf{T}_u \times \mathbf{T}_v)\,du\,dv$$

  does not require $\mathbf{T}_v \times \mathbf{T}_v$ to be unit length
- The equivalent formulation

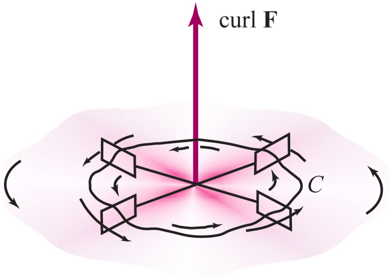
  $$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

  requires $\mathbf{n}$ to be a unit normal vector to S
- This is because

  $$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint \mathbf{F}\cdot\frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|}\,\|\mathbf{T}_u \times \mathbf{T}_v\|\,du\,dv$$

Physical interpretations of line integrals

1. Work done by a force: $W = \int_C \mathbf{F} \cdot d\mathbf{r}$
2. Net current around a loop of wire if $\mathbf{F}$ is an electric field
3. Net circulation (rotation) of a fluid around C



Irrotational vector fields

- If $\nabla \times \mathbf{F}(\mathbf{x}) = \mathbf{0}$ then there is no net rotation of $\mathbf{F}$ at the point $\mathbf{x}$
- If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point, then $\mathbf{F}$ is irrotational
- If $\mathbf{F}$ is irrotational, then $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every closed curve C
- If $\mathbf{F}$ is irrotational, then $\mathbf{F} = \nabla f$ for some potential function f

Path independence

- Suppose $\mathbf{F} = \nabla f$ (such as a gravitational field)
- Parametrize C as $\mathbf{r}(t)$, $a \le t \le b$
- The chain rule implies

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

- This implies path independence

Proof

- The parametrization implies

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\ &= \int_a^b \nabla f(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\,dt \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a))\end{aligned}$$

- Note that $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$

Gradient vector fields are irrotational

- Suppose $\mathbf{F} = \nabla f$
- Then $\nabla \times \mathbf{F} = \mathbf{0}$ because $\nabla \times \nabla f = \mathbf{0}$
- Given any capping surface S, Stokes' theorem implies

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0$$

- If $\nabla \times \mathbf{F} = \mathbf{0}$, then there is no circulation around any closed curve

Path independence implies gradient

- It is also true that if $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the endpoints of C, then $\mathbf{F}=\nabla f$ for some f (see the proof on p. 550)
- Therefore,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r}=0 \quad &\Longleftrightarrow \quad \text{path independence}\\ &\Longleftrightarrow \quad \mathbf{F}=\nabla f\\ &\Longleftrightarrow \quad \nabla\times\mathbf{F}=\mathbf{0}\end{aligned}$$

- We say that $\mathbf{F}$ is a conservative vector field

Important

- A finite number of exceptional points is allowed
- An exceptional point is one where $\mathbf{F}$ or its derivative is not defined
- Example: The gravitational potential field

$$\mathbf{F} = \nabla\left(\frac{GMm\,\mathbf{r}}{\|\mathbf{r}\|}\right)$$

is undefined at $\mathbf{r} = \mathbf{0}$, but otherwise possesses the properties of path independence, zero curl, etc.

Example (#7, p. 559)

- Consider $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
- Is $\mathbf{F}$ a conservative (i.e., gradient) vector field?

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- Consider $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
- Is $\mathbf{F}$ a conservative (i.e., gradient) vector field?
- Compute

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xy & y & z \end{vmatrix} = -x\mathbf{k}$$

- Since $\nabla \times \mathbf{F} \neq \mathbf{0}$, this $\mathbf{F}$ is not conservative

Example (#3, p. 558)

- Let $\mathbf{F} = (2xyz + \sin x)\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$
- A calculation shows $\nabla \times \mathbf{F} = \mathbf{0}$
- Hence $\mathbf{F} = \nabla f$. What is ∇f?
- Must solve the system

$$\frac{\partial f}{\partial x} = 2xyz + \sin x, \quad \frac{\partial f}{\partial y} = x^2z, \quad \frac{\partial f}{\partial z} = x^2y$$

Example (#3, p. 558), continued

- Given $\partial f/\partial y = x^2 z$, we must have
$$f = \int x^2 z \, dy = x^2 zy + \text{terms involving only } x \text{ and } z$$

- This gives
$$\begin{aligned} f &= \int x^2 y \, dy = x^2 zy + g(x,z) \\ &= \int x^2 y \, dz = x^2 yz + h(x,y) \\ &= \int (2xyz + \sin x) \, dx = x^2 yz - \cos x + k(y,z) \end{aligned}$$

Example (#3, p. 558), continued

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- Hence $f(x,y,z) = x^2 yz - \cos x + C$