Stokes' theorem

Eric Kostelich

ARIZONA STATE UNIVERSITY
DEPT. OF MATHEMATICS AND STATISTICS

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Reading for this week

- Sections 8.2, 8.3, 8.4
- Common final: Thursday, May 7 from 7:10–9:00 p.m., PSA 109

Discussion question

- Let S_1 be the cylindrical shell of radius 2 centered about the z axis for $0 \leq z \leq 1$
- Let S_2 be the upper hemisphere of radius 2 centered about the z axis and whose bottom surface is on the xy plane
- Let $\mathbf{F} = \mathbf{r}/\|\mathbf{r}\|$
- Find $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$ and $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$

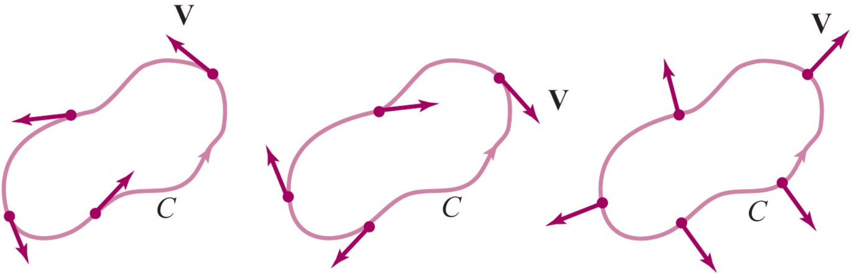
Some remarks about the curl

- The line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\, ds = \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

- $\mathbf{T}$ is always tangent to the path

What the sign says



$\int_C \mathbf{V} \cdot d\mathbf{s} > 0$ $\int_C \mathbf{V} \cdot d\mathbf{s} < 0$ $\int_C \mathbf{V} \cdot d\mathbf{s} = 0$

The meaning of curl in fluid flow

- Suppose $\mathbf{V}$ is the velocity field of a fluid
- Let ∂S be the simple boundary of a small region
- Then
$$\int_{\partial S} \mathbf{V} \cdot d\mathbf{r} = \int_{\partial S} \mathbf{V} \cdot \mathbf{T}\, ds$$
is the net velocity of the fluid around ∂S
- Since
$$\int_{\partial S} \mathbf{V} \cdot d\mathbf{r} = \int_{S} (\nabla \times \mathbf{V}) \cdot \mathbf{n}\, dS$$
the curl of $\mathbf{V}$ is the circulation per unit area on a surface perpendicular to $\mathbf{n}$

Vorticity

- If $\mathbf{V}$ is the velocity field of a fluid, then $\nabla \times \mathbf{V}$ is called the vorticity of the fluid at a point
- Air that rotates counterclockwise has positive vorticity (cyclones, low pressure systems)

Stokes' theorem

- Suppose S is a smooth, orientable surface with boundary ∂S
- Example: capping surface, such as a hemisphere
- Then if ∂S is positively oriented,

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

- Note: If S_1 and S_2 are two different orientable surfaces with the same boundary curve, then

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Example 1

- Consider the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$
- Let $\mathbf{F} = (-y^2, x, z^2)$
- Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Example 1

- Consider the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y + z = 2$
- Let $\mathbf{F} = (-y^2, x, z^2)$
- Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$.
- Solution: By Stokes' theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot (\mathbf{T}_u \times \mathbf{T}_v) \, du \, dv$$

Example 1, continued

- Here $\mathbf{T}_y \times \mathbf{T}_z = (0,1,1)$ is a normal vector to the surface
- A calculation shows $\nabla \times (-y^2, x, z^2) = (1+2y)\mathbf{k}$
- Therefore,

$$\begin{aligned} \oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F})\cdot d\mathbf{S} \\ &= \iint_D (1+2y)\, dA, \end{aligned}$$

where D is the unit circle.

Example 1, continued

- This gives

$$\begin{aligned}\iint_D (1+2y)\,dA &= \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)r\,dr\,d\theta \\ &= \int_0^{2\pi} (\tfrac{1}{2}+\tfrac{2}{3}\sin\theta)\,d\theta \\ &= \pi.\end{aligned}$$

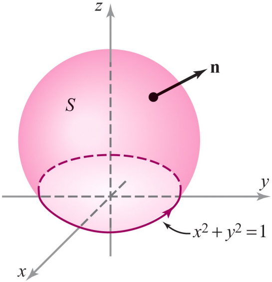
Example 1, continued

- Alternative approach: Evaluate the line integral directly
- Since $y+z=2$, one parametrization of the curve of intersection is $\mathbf{r}(t) = (\cos t, \sin t, 2-\sin t)$
- Since $\mathbf{F} = (-y^2, x, z^2)$, we have

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin^2 t, \cos t, (2-\sin t)^2)\cdot \\
&\qquad (-\sin t, \cos t, -\cos t)\, dt \\
&= \int_0^{2\pi} (\sin^3 t + \cos^2 t - (\cos t)(2-\sin t)^2)\, dt \\
&= 0 + \pi + 0.
\end{aligned}$$

Example 3, p. 538

- Suppose S is a surface as indicated
- Let $\mathbf{F} = (y, -x, e^{xz})$
- Compute $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$



Stokes' theorem to the rescue

- No need to find a parametrization of S
- Since ∂S is the unit circle, evaluate the line integral:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_0^{2\pi} (\sin t, -\cos t, 0) \cdot (-\sin t, \cos t, 0)\, dt$$