

Review of line and surface integrals

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Reading for this week

- Sections 7.6, 8.1, 8.2

Overview of integrals we have discussed so far

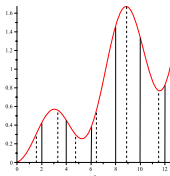
- Path integral: $\int_C f \, ds$
- Line integral: $\int_C \mathbf{F} \cdot d\mathbf{r}$
- Surface integral: $\iint_S f \, dS$
- Flux integral: $\iint_S \mathbf{F} \cdot d\mathbf{S}$

The Riemann integral is the limit of sums

- Approximate the area under the curve $f(x)$ by rectangles:

$$\sum_{i=1}^n f(x_i) \Delta x \approx \sum_{i=1}^n \text{height} \times \text{width of } i\text{th rectangle}$$

- The interval $[a, b]$ is divided into n equal subintervals
- Pick a point x_i from the i th subinterval



Riemann integral, continued

- If the sums converge to a limiting value as $n \rightarrow \infty$, then we say that f is **Riemann integrable** on $[a, b]$ and write

$$\int_a^b f(x) dx$$

to represent the limit.

- The **fundamental theorem of calculus** says that if $F' = f$ on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Path integrals

- We can apply the same idea to a path instead of an interval
- Divide C into n equal pieces of length Δs
- Evaluate f at some point \mathbf{x}_i on the i th piece
- Consider the sum

$$\sum_{i=1}^n f(\mathbf{x}_i) \Delta s$$

- If this process converges, we denote the limit as $\int_C f ds$

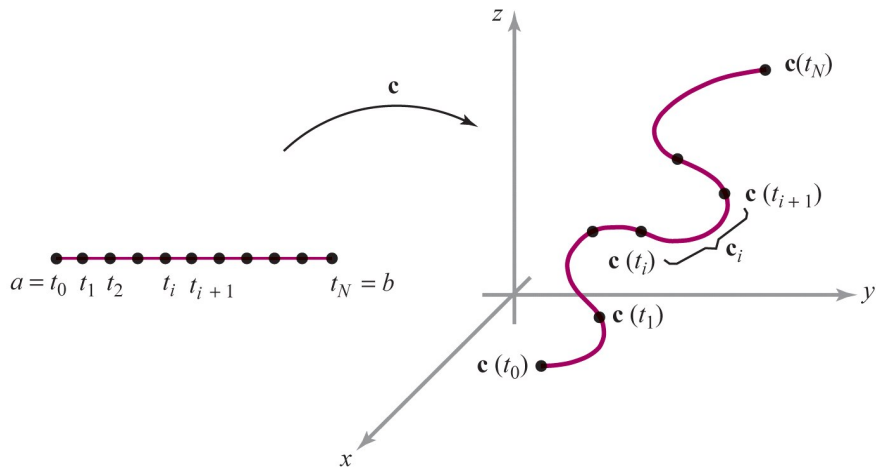
Physical motivation

- Suppose f is the mass or charge density per unit length along a piece of wire
- Then $\sum_{i=1}^n f(\mathbf{x}_i) \Delta s$ approximates the total mass or charge
- To make the calculation practical, we use a parametrization $\mathbf{r}(t)$, $a \leq t \leq b$ of the curve in question
- Then

$$\sum_{i=1}^n f(\mathbf{x}_i) \Delta s = \sum_{i=1}^n f(\mathbf{r}(t_i)) \Delta s_i$$

where $\Delta s_i = \|\mathbf{r}'(t_i)\| \Delta t$

Sketch of the process



The path integral

- If

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(\mathbf{r}(t_i)) \Delta s_i$$

exists, then we write

$$\int_C f ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt$$

for the parametrization $\mathbf{r}(t)$

- Entirely analogous to the usual Riemann integral

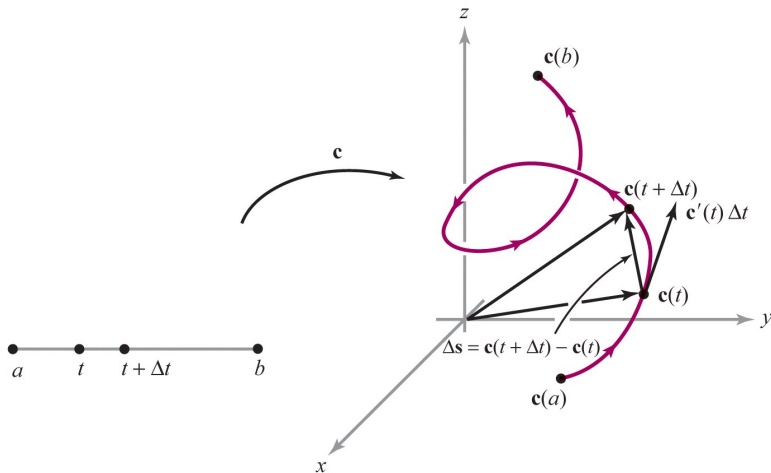
Line integral

- **Physical motivation:** $W = \mathbf{F} \cdot \mathbf{d}$
- Suppose that \mathbf{F} is not constant
- Divide the path into n pieces
- Then $\Delta W \approx \mathbf{F}(\mathbf{x}_i) \Delta \mathbf{s}_i$ where $\Delta \mathbf{s}_i = \mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i)$
- The total work is

$$W \approx \sum_{i=1}^n \mathbf{F}(\mathbf{x}_i) \Delta \mathbf{s}_i$$

- If this process converges, we define the limit as $\int_C \mathbf{F} \cdot d\mathbf{r}$

Schematic diagram of this process



Key approximation

- $\Delta \mathbf{s}_i = \mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i) \approx \mathbf{r}'(t_i) \Delta t$
- Becomes exact as $\Delta t \rightarrow 0$ (i.e., as $n \rightarrow \infty$)
- We compute

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

Connection with path integrals

- Let $\mathbf{r}(t)$ be a parametrization of the curve C
- The **unit tangent vector** is $\mathbf{T}(t) = \mathbf{r}'(t) / \|\mathbf{r}'(t)\|$
- Then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}(t) \|\mathbf{r}'(t)\| dt \\ &= \int_C (\mathbf{F} \cdot \mathbf{T}) ds.\end{aligned}$$

Surface integrals

- Similar physical motivation: mass or charge density per unit area
- Divide the surface up into n patches of area ΔA_i , $i = 1, \dots, n$
- Consider $\sum_{i=1}^n f(\mathbf{x}_i) \Delta A_i$ where \mathbf{x}_i is a point on the i th patch
- If the sums converge to a limit as $n \rightarrow \infty$, then we define the limit as $\iint_S f \, dS$.

The trick is the parametrization

- Parametrize S as $\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$
- Define the tangent vectors $\mathbf{T}_u = \partial\Phi/\partial u$ and $\mathbf{T}_v = \partial\Phi/\partial v$
- A rectangle of dimensions $\Delta u \Delta v$ centered at (u, v) maps to a piece of surface of area approximately $\|\mathbf{T}_u \times \mathbf{T}_v\| \Delta u \Delta v$ centered at $\Phi(u, v)$

The parametrization gives the integral

- Let $\Phi(u, v)$ be a parametrization of the surface S for $(u, v) \in [a, b] \times [c, d]$
- Assume that S is bounded, smooth (regular), and orientable

- Then

$$\iint_S f \, dS = \int_c^d \int_a^b f(\Phi(u, v)) \|\mathbf{T}_u \times \mathbf{T}_v\| \, du \, dv$$

- Can be hard to evaluate but the simple cases are the most important

Connection to normal vectors and flux

- $\mathbf{a} \times \mathbf{b}$ is normal to both \mathbf{a} and \mathbf{b}
- $\mathbf{T}_u \times \mathbf{T}_v$ is normal to S at $\Phi(u, v)$
- The **flux** of the vector field \mathbf{F} to S at \mathbf{x} is $\mathbf{F}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$ where \mathbf{n} is the normal vector to S at \mathbf{x}
- Must specify whether \mathbf{n} is inward or outward pointing
- Divide S into n patches as before and consider

$$\sum_{i=1}^n \mathbf{F}(\mathbf{x}_i) \cdot \mathbf{n}(\mathbf{x}_i) \Delta A_i$$

- If $\lim_{n \rightarrow \infty}$ exists we write $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \mathbf{F} \cdot d\mathbf{S}$

The parametrization is key

- Given $\Phi(u, v)$, $\mathbf{T}_u = \partial\Phi/\partial u$, $\mathbf{T}_v = \partial\Phi/\partial v$
- Then, at each point $\Phi(u, v)$ of S , a unit normal is

$$\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|}$$

and $dS = \|\mathbf{T}_u \times \mathbf{T}_v\| du dv$

- Hence

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_c^d \int_a^b \mathbf{F}(\Phi(u, v)) \cdot \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} dS \\ &= \int_c^d \int_a^b \mathbf{F}(\Phi(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) du dv \end{aligned}$$

Know the important simplifications

- The evaluation of a surface integral can be difficult
- But certain common cases are simple!
- For a sphere of radius ρ centered at the origin,
 $\mathbf{n} = (x, y, z)/\rho$
- For a cylinder of radius r centered at the z axis,
 $\mathbf{n} = (x, y, 0)/r$

Discussion questions

- 1 Compute $\iint_S z \, dS$ where S is the cylinder of radius 2 centered on the z axis, $0 \leq z \leq 1$
- 2 Compute $\iint_S z \, dS$ where S is the portion of the sphere of radius 2 centered at the origin that lies above the xy plane
- 3 Compute $\int_C z \, ds$ where C is an arc extending from the equator of the sphere in Problem 2, going across the north pole, then back to the equator 180° opposite