Green's Theorem

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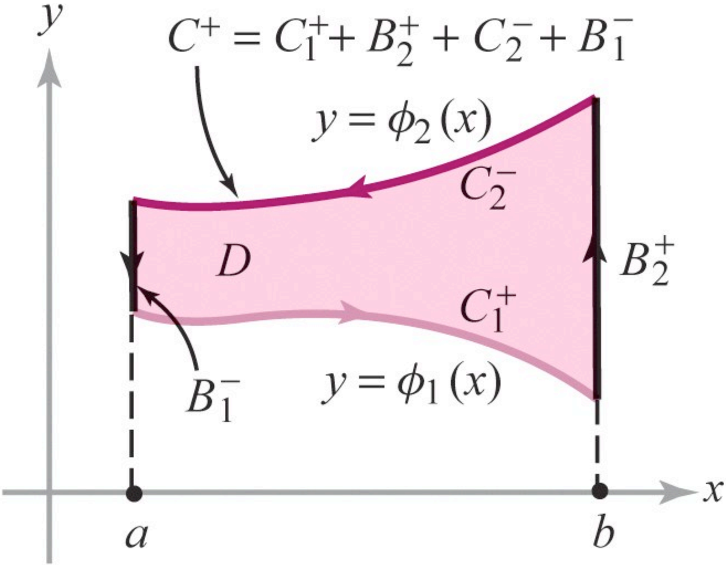
Reading for this week

- Sections 7.6, 8.1, 8.2

Review: Positive boundary orientation on a y-simple region



Green's theorem: special case

- Lemma 1: Suppose D is y-simple and that $\mathbf{F} = P\mathbf{i} + 0\mathbf{j}$ is a smooth vector field. Then

$$\int_{\partial D^+} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D^+} P\, dx = -\iint_D \frac{\partial P}{\partial y}\, dA.$$

- Proof: Fubini's theorem implies

$$\begin{aligned} \iint_D \frac{\partial P}{\partial y}\, dA &= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}\, dy\, dx \\ &= \int_a^b \left[P(x,\phi_2(x)) - P(x,\phi_1(x))\right] dx. \end{aligned}$$

Consider the lower boundary C_1^+

- A natural parametrization is $\mathbf{r}(t) = (t, \phi_1(t))$ for $a \le t \le b$
- Since $\mathbf{F} = P\mathbf{i} + 0\mathbf{j}$, we have

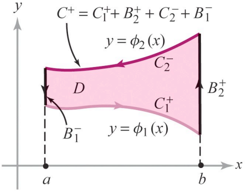
$$\begin{aligned} \int_{C_1^+} (P,0) \cdot d\mathbf{r} &= \int_a^b (P,0) \cdot (1, \phi_1') \, dt \\ &= \int_a^b P(t, \phi_1(t)) \, dt \end{aligned}$$

- Hence $-\iint_D \frac{\partial P}{\partial y} \, dA = \int_{C_1^+} -P \, dx$

The top boundary C_2^-

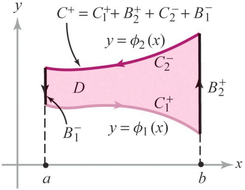
- Traverse from right to left: $\mathbf{r}(t) = (t, \phi_2(t))$, $b \geq t \geq a$
- So

$$\int_{C_2^-} (P, 0) \cdot d\mathbf{r} = \int_b^a (P, 0) \cdot (1, \phi_2') \, dt$$
$$= -\int_a^b P(t, \phi_2(t)) \, dt.$$



The sides

- The right side is parametrized as $\mathbf{r}(t) = (b,t)$, $\phi_1(b) \le t \le \phi_2(b)$
- Therefore, $\displaystyle\int_{B_2^+} (P,0)\cdot d\mathbf{r} = \int_{\phi_1(b)}^{\phi_2(b)} (P,0)\cdot(0,1)\,dt = 0$
- Similar situation for the left side, which is parametrized as $\mathbf{r}(t) = (a,t)$, $\phi_2(a) \ge t \ge \phi_1(a)$



Put this all together

- From Fubini's theorem:

$$\iint_D \frac{\partial P}{\partial y}\, dA = \int_a^b \left[P(x,\phi_2(x)) - P(x,\phi_1(x)) \right] dx$$

- The line integrals: Sides contribute nothing. Top and bottom give

$$\begin{aligned} \int_{C_2^-} (P,0)\cdot d\mathbf{r} &= -\int_a^b P(t,\phi_2(t))\, dt \\ \int_{C_1^+} (P,0)\cdot d\mathbf{r} &= \int_a^b P(t,\phi_1(t))\, dt \end{aligned}$$

Lemma 1, concluded

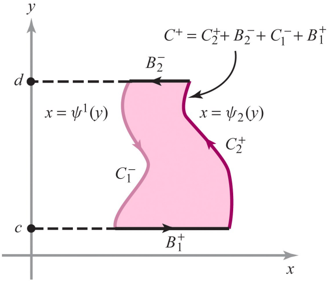
- Therefore,

$$-\iint_D \frac{\partial P}{\partial y}\, dA = \int_{\partial D^+} P\, dx$$

Green's theorem: second special case

- Lemma 2: Let $\mathbf{F} = 0\mathbf{i} + Q\mathbf{j}$ be smooth and D an x-simple boundary. Then

$$\int_{\partial D^+} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D^+} Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA.$$



Lemma 2, continued

- Similar argument: top and bottom contribute nothing.
- Right side (C_2^+): $\mathbf{r}(t) = (\psi_2(t), t)$, $c \leq t \leq d$
- Then

$$\begin{aligned} \int_{C_2^+} \mathbf{F} \cdot d\mathbf{r} &= \int_c^d (0, Q(\mathbf{r})) \cdot (\psi_2', 1)\, dt \\ &= \int_c^d Q(\psi_2(t), t)\, dt. \end{aligned}$$

Lemma 2, continued

- Fubini's theorem gives

$$\iint_D \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy$$

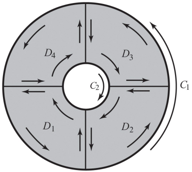
$$= \int_c^d \left[Q(\psi_2(y),y) - Q(\psi_1(y),y)\right]dy$$

$$= \int_{C_2^+ \cup C_1^-} Q\,dy.$$

- Theorem: Suppose D is a simple region with boundary ∂D, and suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a smooth vector field. Then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_{\partial D^+} P\, dx + Q\, dy = \int_{\partial D^+} \mathbf{F} \cdot d\mathbf{r}$$

- Note: Green's theorem is also valid for any connected region in $\mathbb{R}^2$



Example (#3a, p. 528)

- Let D be the disk of radius R centered at the origin
- Verify Green's theorem for $\mathbf{F} = (xy^2, -yx^2)$
- Solution:

$$\begin{aligned}
\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dAy &= \iint_D \left(\frac{\partial}{\partial x}(-yx^2) - \frac{\partial}{\partial y} xy^2 \right) \, d \\
&= \iint_D -4xy \, dA \\
&= \int_0^{2\pi} \int_0^R -4r^3 \cos\theta \sin\theta \, dr \, d \\
&= 0.
\end{aligned}$$

Example #3a, continued

- The line integral is

$$
\begin{aligned}
\int_{\partial D^+} \mathbf{F} \cdot d\mathbf{r} &= \int_{\partial D^+} (xy^2, -yx^2) \cdot d\mathbf{r} \\
&= \int_0^{2\pi} (R^3 \cos\theta \sin^2\theta, -R^3 \sin\theta \cos^2\theta) \\
&= \qquad \cdot (-R\sin\theta, R\cos\theta)\, d\theta \\
&= \int_0^{2\pi} R^4 (-\cos\theta \sin^3\theta + \sin\theta \cos^3\theta)\, d\theta \\
&= 0.
\end{aligned}
$$

Discussion questions

1. (#3b, p. 528) Verify Green's theorem for the disk D of radius R for the vector field $\mathbf{F} = (x+y, y)$
2. Find the area of D using Theorem 2:

$$A = \frac{1}{2} \oint_{\partial D^+} x\,dy - y\,dx.$$