Green's Theorem (and some review)

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Reading for this week

- Sections 7.6, 8.1, 8.2

Some remarks on surface integrals

- The region of integration depends on the surface
- Example: The half of the ellipsoid $x^2 + y^2 + 3z^2 = 1$ below the xy plane
- We map the unit disk to the surface, so that's the region of integration (use polar coordinates)
- Example: The usual spherical coordinates map the rectangle $\{(\theta, \phi) : 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi\}$ to the sphere
- But this mapping is not 1-to-1 at the poles!
- Notice that $\|\mathbf{T}_\theta \times \mathbf{T}_\phi\| = \rho^2 \sin\phi$ vanishes there

Unit normals

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- What is a unit normal to the cylinder of radius r centered on the z axis?

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- Yesterday's lab shows that a unit normal to the sphere of radius ρ is $(x, y, z)/\rho$
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- Answer: $(x, y, 0)/r$
- Moral of story: use the coordinates that simplify the problem the most

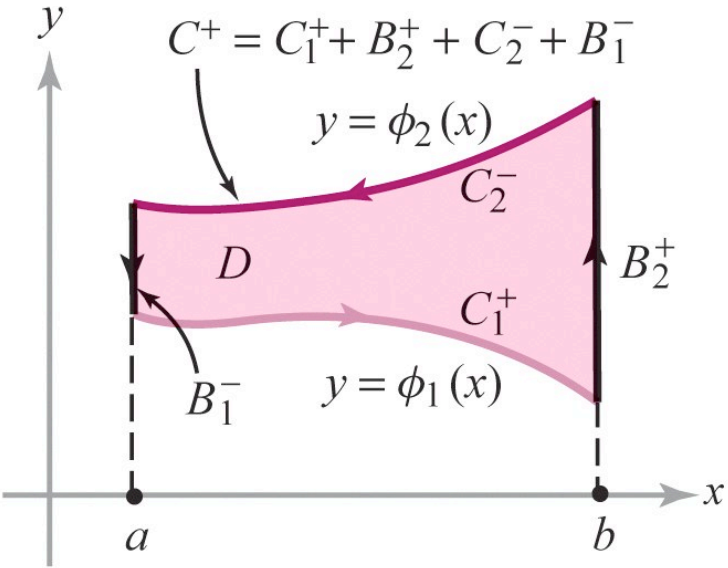
Green's Theorem: Introduction

- One-dimensional integrals are evaluated by relating a function over an interval to another on the endpoints
- Example: Given $f(x) = 3x^2$, we consider the antiderivative $F(x) = x^3$
- Then $\int_a^b f(x)\,dx = F(b) - F(a)$
- For instance, $\int_1^2 3x^2\,dx = 2^3 - 1^3 = 7$

Green's, Stokes', and Gauss's Theorems

- The great theorems of vector calculus are generalizations of this idea
- We relate quantities evaluated over a surface S to something evaluated on the boundary ∂S
- This often simplifies the computation of flux integrals
- Green's Theorem relates the computation of a double integral over the enclosed region to a line integral on the boundary

Positive boundary orientation on a *y*-simple region



Green's theorem: special case

- Lemma: Suppose D is y-simple and that $\mathbf{F} = P\mathbf{i} + 0\mathbf{j}$ is a smooth vector field. Then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA.$$

- Proof: Fubini's theorem implies

$$\begin{aligned} \iint_D \frac{\partial P}{\partial y}\,dA &= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}\,dy\,dx \\ &= \int_a^b \left[P(x,\phi_2(x)) - P(x,\phi_1(x))\right] dx. \end{aligned}$$

Consider the lower boundary C_1^+

- A natural parametrization is $\mathbf{r}(t) = (t, \phi_1(t))$ for $a \le t \le b$
- Now

$$\begin{aligned}\int_{C_1^+} (P,0)\cdot d\mathbf{r} &= \int_a^b (P,0)\cdot(1,\phi_1')\,dt \\ &= \int_a^b P(t,\phi_1(t))\,dt\end{aligned}$$