Flux integrals

Eric Kostelich

ARIZONA STATE UNIVERSITY
DEPT. OF MATHEMATICS AND STATISTICS

April 13, 2009

Reading for this week

- Sections 7.6, 8.1, 8.2

Surface normals

- $\mathbf{T}_u \times \mathbf{T}_v$ is normal to the surface S at every point
- Suppose $\mathbf{F}$ is a velocity field
- If $\mathbf{n}$ is a unit normal, then $\mathbf{F} \cdot \mathbf{n}$ is the flux across S at a given point
- If the flow is parallel to S (i.e., perpendicular to $\mathbf{n}$) then there is no flux across S at that point

Total flux

- Definition: The total flux of $\mathbf{F}$ across the surface S is

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

- Assumes that S is orientable and smooth (i.e., $\mathbf{n}$ is well defined everywhere)
- A parametrization of S is necessary to evaluate the integral

Evaluating flux integrals

- Find a parametrization $\Phi(u, v)$ of S
- Compute $\mathbf{T}_u \times \mathbf{T}_v$ at each point
- Now

$$\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} \quad \text{and} \quad dS = \|\mathbf{T}_u \times \mathbf{T}_v\| \, du \, dv$$

- Hence

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v) \, du \, dv$$

Some simplifications

- For a sphere of radius ρ centered at the origin, an outward normal vector is $\rho\mathbf{r} = \rho(x, y, z)$
- The outward unit normal is $\mathbf{r}$
- If $\mathbf{F} = \mathbf{r} = (x, y, z)$ and $\rho = 1$, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (x, y, z) \cdot \mathbf{r}\, dS = \iint_S 1\, dS = 4\pi.$$

Other considerations

- If S consists of two distinct pieces, then calculate the flux across each piece separately and add
- Example: Suppose S is the surface that encloses the hemisphere of radius 1 above the xy plane together with the unit circle
- Consider $S = H \cup D$
- Outward unit normals: $\mathbf{r}$ for H and $-\mathbf{k}$ for D

Example (#6, p. 495)

- Consider $z = 12$ and $x^2 + y^2 = 25$ (a disk D of radius 5)
- Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Compute $\iint_D \mathbf{r} \cdot d\mathbf{S}$.

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- Solution:

$$\iint_D \mathbf{r} \cdot d\mathbf{S} = \iint_D \mathbf{r} \cdot \mathbf{n}\, dS = \iint_D z\, dS = 12 \times \text{Area}(D)$$

Discussion questions

1. Let S be the upper half of the sphere of radius 2 centered at the origin. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = (2x, 2y, 2z)$.
2. Let $\mathbf{F} = (2x, 2y, 2z)$ and compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where S is the cylinder of radius 1 and height 1 centered about the z axis $(0 \leq z \leq 1)$

Answer to Problem 1

- The sphere of radius ρ has surface area $4\pi\rho^2$
- On the sphere, $\mathbf{n} = \mathbf{r} = (x, y, z)$
- Here $\mathbf{F} = 2\mathbf{r}$ so $\mathbf{F} \cdot \mathbf{n} = 2\mathbf{r} \cdot \mathbf{r} = 2\|\mathbf{r}\|^2 = 8$
- Hence $\iint_{\text{hemisphere}} \mathbf{F} \cdot d\mathbf{S} = 8 \times (2\pi \cdot 2^2) = 32\pi$
- On the disk, $\mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot (-\mathbf{k}) = -z = 0$
- So the net flux is 32π

Answer to Problem 2

- Top: $\iint_T \mathbf{F} \cdot \mathbf{n}\, dS = 2z \times \text{Area(top)} = 2\pi$
- Bottom: $\iint_B \mathbf{F} \cdot \mathbf{n}\, dS = -2z \times \text{Area(bottom)} = 0$
- Sides: $\mathbf{n} = (x, y, 0)$ so
  $\iint_B \mathbf{F} \cdot \mathbf{n}\, dS = (x^2 + y^2) \times \text{Area(sides)} = 2\pi$
- Hence $\iint_{\text{cylinder}} \mathbf{F} \cdot d\mathbf{S} = 4\pi$

Special case: graphs

- Suppose that the surface S is the graph of a function: $z = g(x, y)$
- Then S may be parametrized as $\Phi(x, y) = (x, y, g(x, y))$
- This gives

$$\begin{aligned}
\mathbf{T}_x &= (1, 0, \partial g/\partial x) \\
\mathbf{T}_y &= (0, 1, \partial g/\partial y) \\
\mathbf{T}_x \times \mathbf{T}_y &= (-\partial g/\partial x, -\partial g/\partial y, 1)
\end{aligned}$$

Example

- Let S be the section of the paraboloid determined by $z = -x^2 - y^2$, $-1 \leq z \leq 0$ with $\mathbf{n}$ the upward (outward) pointing normal.
- Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = (x, y, z)$

Example

- Let S be the section of the paraboloid determined by $z = -x^2 - y^2$, $-1 \leq z \leq 0$ with $\mathbf{n}$ the upward (outward) pointing normal.
- Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = (x, y, z)$
- Solution: S is the image of the unit disk D under the parametrization $\Phi(x, y, -x^2 - y^2)$
- Since S is a graph, $\mathbf{T}_x \times \mathbf{T}_y = (2x, 2y, 1)$ (upward/outward)

Example, continued

- Then

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D (x,y,z)\cdot(2x,2y,1)\,dx\,dy \\ &= \iint_D 3(x^2+y^2)\,dx\,dy \\ &= \int_0^{2\pi}\int_0^1 3r^3\,dr\,d\theta \\ &= 3\pi/2.\end{aligned}$$