Flux integrals

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Reading for this week

- Section 7.6
- Next exam: Friday, April 10

Parametrized surfaces

- The parametrization function has the form

$$\Phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$

- $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$
- Define the tangent vectors

$$\begin{aligned} \mathbf{T}_u &= \partial\Phi/\partial u \\ \mathbf{T}_v &= \partial\Phi/\partial v \end{aligned}$$

- If $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$ at a given (u_0, v_0), then we say that Φ is smooth there

Surface normals

- $\mathbf{T}_u \times \mathbf{T}_v$ is normal to the surface S at every point
- Suppose $\mathbf{F}$ is a velocity field
- If $\mathbf{n}$ is a unit normal, then $\mathbf{F} \cdot \mathbf{n}$ is the flux across S at a given point
- If the flow is parallel to S (i.e., perpendicular to $\mathbf{n}$) then there is no flux across S at that point

Total flux

- Definition: The total flux of $\mathbf{F}$ across the surface S is

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

- Assumes that S is orientable and smooth (i.e., $\mathbf{n}$ is well defined everywhere)
- A parametrization of S is necessary to evaluate the integral

Evaluating flux integrals

- Find a parametrization $\Phi(u, v)$ of S
- Compute $\mathbf{T}_u \times \mathbf{T}_v$ at each point
- Now

$$\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|} \quad \text{and} \quad dS = \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$$

- Hence

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv$$

Orientation

- A surface is orientable if it has an “inside” and an “outside”
- A Möbius band is not orientable
- Example: The unit sphere has an outward and an inward normal surface vector
- Using the opposite normal gives the negative of the original integral

Computations (from last time)

- Note that

$$\begin{aligned}
\mathbf{T}_u \times \mathbf{T}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \\
&= \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \mathbf{k} \\
&= \left(\frac{\partial(y,z)}{\partial(u,v)} \right) \mathbf{i} - \left(\frac{\partial(x,z)}{\partial(u,v)} \right) \mathbf{j} + \left(\frac{\partial(x,y)}{\partial(u,v)} \right) \mathbf{k}
\end{aligned}$$

The normal to the sphere

- Given the parametrization

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

we have

$$\begin{aligned} \mathbf{T}_\theta &= (-\rho \sin\phi \sin\theta, \rho \sin\phi \cos\theta, 0) \\ \mathbf{T}_\phi &= (\rho \cos\phi \cos\theta, \rho \cos\phi \sin\theta, -\rho \sin\phi) \\ \mathbf{T}_\theta \times \mathbf{T}_\phi &= -\rho^2 \sin\phi (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi) \end{aligned}$$

- Does this normal vector point inward or outward?

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- Does this normal vector point inward or outward?
- Inward—so Φ is an orientation reversing parametrization

Example (#1, pp. 484–485)

- Suppose $\mathbf{F} = (x, y, z)$. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where S is the unit sphere with the previous parametrization
- Answer: Since $\rho = 1$, we have

$$\begin{aligned} \mathbf{F} \cdot (\mathbf{T}_\theta \times \mathbf{T}_\phi) &= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \cdot \\ &\quad -\sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \\ &= -\sin\phi. \end{aligned}$$

- Therefore

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} -\sin\phi \; d\phi \; d\theta = -4\pi.$$

Example

- Suppose the problem specifies an outward oriented normal
- Then we take

$$\mathbf{T}_\theta \times \mathbf{T}_\phi = +\rho^2 \sin\phi\,(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

- With this orientation and $\rho = 1$, we have

$$\iint_S (x,y,z)\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} +\sin\phi\; d\phi\; d\theta = +4\pi.$$

Some simplifications

- For a sphere of radius ρ centered at the origin, an outward normal vector is $\rho\mathbf{r} = \rho(x, y, z)$
- The outward unit normal is $\mathbf{r}$
- If $\mathbf{F} = \mathbf{r} = (x, y, z)$ and $\rho = 1$, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (x, y, z) \cdot \mathbf{r} \, dS = \iint_S 1 \, dS = 4\pi.$$

Other considerations

- If S consists of two distinct pieces, then calculate the flux across each piece separately and add
- Example: Suppose S is the surface that encloses the hemisphere of radius 1 above the xy plane together with the unit circle
- Consider $S = H \cup D$
- Outward unit normals: $\mathbf{r}$ for H and $-\mathbf{k}$ for D

Discussion questions

1. Let $S = H \cup D$ as above. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = (2x, 2y, 2z)$.
2. Let $\mathbf{F} = (2x, 2y, 2z)$ and compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where S is the cylinder of radius 1 and height 1 centered about the z axis ($0 \leq z \leq 1$)