Surface area and surface integrals

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Reading for this week

- Sections 7.1–7.4
- Next exam postponed until Friday, April 10
- Homework is posted and is due next Friday

Parametrized surfaces

- The parametrization function has the form

$$\Phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$

- $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$
- Define the tangent vectors

$$\begin{aligned} \mathbf{T}_u &= \partial\Phi/\partial u \\ \mathbf{T}_v &= \partial\Phi/\partial v \end{aligned}$$

- If $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$ at a given (u_0, v_0), then we say that Φ is smooth there

Area of a surface

- Parametrize the smooth surface S as $\Phi(u, v)$ for $a \leq u \leq b$, $c \leq v \leq d$
- Define $\mathbf{T}_u = \partial\Phi/\partial u$ and $\mathbf{T}_v = \partial\Phi/\partial v$
- Definition: The surface area of S is

$$A(S) = \int_a^b \int_c^d \|\mathbf{T}_u \times \mathbf{T}_v\| \, dv \, du.$$

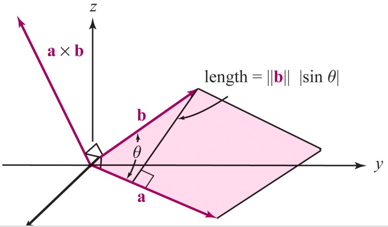
Important notes

- Assume that S is smooth except possibly at a finite number of points
- S can be a "quilt", i.e., the union of a finite number of smaller surfaces
- In this case we require smoothness except at the corners of the patches in the quilt

Justification for the area formula

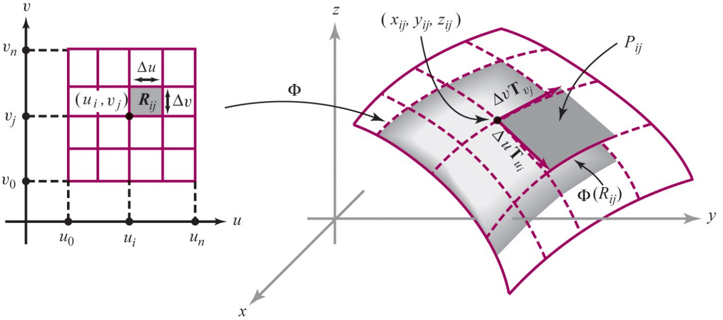
- In $\mathbb{R}^3$, the area of the parallelogram spanned by vectors $\mathbf{a}$ and $\mathbf{b}$ is $\|\mathbf{a} \times \mathbf{b}\|$
- Proof: Some algebra shows that

$$\begin{aligned} \|\mathbf{a} \times \mathbf{b}\|^2 &= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 (1 - \cos^2 \theta) \\ &= \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \sin^2 \theta \end{aligned}$$



Justification, 2

- A little box $\Delta u \Delta v$ at (u_i, v_j) gets mapped by Φ to a little parallelogram of area $\|\mathbf{T}_u \Delta u \times \mathbf{T}_v \Delta v\|$ where the derivatives are evaluated at (u_i, v_j)



Justification, 3

- Sum up over all the little patches:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \|(\mathbf{T}_u \times \mathbf{T}_v)(u_i, v_j)\| \, \Delta u \Delta v$$

- Take limits as $\Delta u \to 0$ and $\Delta v \to 0$
- Then

$$A(S) = \iint_D \|\mathbf{T}_u \times \mathbf{T}_v\| \, du \, dv$$

- Note that

$$\begin{aligned}
\mathbf{T}_u \times \mathbf{T}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \\
&= \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \mathbf{k} \\
&= \left(\frac{\partial(y,z)}{\partial(u,v)} \right) \mathbf{i} - \left(\frac{\partial(x,z)}{\partial(u,v)} \right) \mathbf{j} + \left(\frac{\partial(x,y)}{\partial(u,v)} \right) \mathbf{k}
\end{aligned}$$

Caveats

- The mapping Φ must be "mostly" 1-to-1
- Example: The sphere of radius ρ may be parametrized as

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

- But if we take $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$ then we cover the sphere twice! (Need $0 \leq \phi \leq \pi$ instead)
- Φ is not 1-to-1 when $\phi = 0$ or $\theta = 2\pi$ but overlaps are a point or curve with no area

The surface area of the sphere

- Given the parametrization

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

we have

$$\begin{aligned}
\mathbf{T}_\theta &= (-\rho \sin\phi \sin\theta, \rho \sin\phi \cos\theta, 0) \\
\mathbf{T}_\phi &= (\rho \cos\phi \cos\theta, \rho \cos\phi \sin\theta, -\rho \sin\phi) \\
\mathbf{T}_\theta \times \mathbf{T}_\phi &= -\rho^2 \sin\phi (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi) \\
\|\mathbf{T}_\theta \times \mathbf{T}_\phi\| &= \rho^2 \sin\phi
\end{aligned}$$

- Then $SA = \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin\phi \; d\phi \; d\theta = 4\pi\rho^2$

Surface integrals

- Given a parametrization $\mathbf{r}(t)$, $a \leq t \leq b$ of a curve C, we define the path integral

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\,\|\mathbf{r}'(t)\|\,dt$$

- Given a parametrization $\Phi(u,v)$, $(u,v) \in [a,b] \times [c,d]$ of a surface S, we define the surface integral

$$\iint_S f\,dS = \int_c^d \int_a^b f(\Phi(u,v))\,\|\mathbf{T}_u \times \mathbf{T}_v\|\,du\,dv$$

Example

- What is the mass of a spherical shell of radius ρ whose mass density at (θ, ϕ) is $\mu = \sin\phi$?
- The total mass is

$$\begin{aligned} \iint_S \mu \, dS &= \iint_S \mu(\theta,\phi)\,\|\mathbf{T}_\theta \times \mathbf{T}_\phi\|\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}(\sin\phi)(\rho^2\sin\phi)\,d\phi\,d\theta \\ &= (\pi\rho)^2 \end{aligned}$$

Note: $\sin^2\phi = (1-\cos 2\phi)/2$

Discussion questions

1. Evaluate $\iint_S z\,dS$, where S is the upper half of the hemisphere of radius a
2. One parametrization of the torus of radius $R > 1$ is

   $$x = (R + \cos\phi)\cos\theta, \quad y = (R + \cos\phi)\sin\theta, \quad z = \sin\phi$$

   for $(\theta, \phi) \in [0, 2\pi] \times [0, 2\pi]$. Find the surface area.

- Spherical coordinates: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$
- $\|\mathbf{T}_\theta \times \mathbf{T}_\phi\| = \rho^2\sin\phi$

Answers

- The sphere:

$$\iint_S z\, dS = \int_0^{2\pi} \int_0^{\pi/2} (a\cos\phi)(a^2 \sin\phi)\, d\phi\, d\theta = \pi a^3$$

- The torus:

$$SA = \int_0^{2\pi} \int_0^{2\pi} (R + \cos\phi)\, d\phi\, d\theta = 4\pi^2 R$$