More on Line Integrals and Parametrized Surfaces

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April 1, 2009

Reading for this week

- Sections 7.1–7.4
- Next exam postponed until Friday, April 10
- Next homework will be posted Friday and due next Friday

Summary: path and line integrals

- Parametrize the curve C as $\mathbf{r}(t)$, $a \le t \le b$.
- Path integral for a scalar function f:

$$\int_C f\, ds = \int_a^b f(\mathbf{r}(t))\, \|\mathbf{r}'(t)\|\, dt$$

- Line integral for a vector field $\mathbf{F}$:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

Equivalent notations for line integrals

- Let $\mathbf{F} = (P, Q, R)$ and $\mathbf{c}(t)$ be a parametrization of the curve C for $a \le t \le b$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_C P\, dx + Q\, dy + R\, dz$$

- The rightmost integral is in differential form

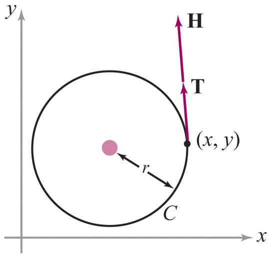
Another connection between path and line integrals

- Let $\mathbf{r}(t)$ be a parametrization of the curve C
- The unit tangent vector is $\mathbf{T}(t) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|$
- Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{T}(t)\, \|\mathbf{r}'(t)\|\, dt \\ &= \int_C (\mathbf{F} \cdot \mathbf{T})\, ds. \end{aligned}$$

Ampère's Law

- Suppose a current I flows along a straight wire
- Then I induces a magnetic field $\mathbf{H}$
- Ampère's Law: In appropriate units, $\oint_C \mathbf{H} \cdot d\mathbf{r} = I$
- $\|\mathbf{H}\|$ is constant on any circle centered on the wire



Review: the chain rule

- Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable
- Let $\mathbf{r}(t)$ be a differentiable path in $\mathbb{R}^3$
- Then

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

Gradient vector fields

- A gradient vector field is $\mathbf{F} = \nabla V$ where V is a scalar-valued function called a potential function
- Example: The gravitational potential

$$V(\mathbf{r}) = \frac{GMm}{\|\mathbf{r}\|} \quad \text{where} \quad \mathbf{r} = (x, y, z)$$

- The gravitational force is

$$\mathbf{F} = \nabla V = -\frac{GMm}{\|\mathbf{r}\|^3}\,\mathbf{r}.$$

Work and path independence

- Motivation: Line integrals represent the work done by the vector field $\mathbf{F}$ as a particle moves along the path $\mathbf{r}(t)$
- When does the work done depend only on the endpoints and not on the path in between?

Work and path independence

- Motivation: Line integrals represent the work done by the vector field $\mathbf{F}$ as a particle moves along the path $\mathbf{r}(t)$
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- Answer: When $\mathbf{F}$ is a gradient vector field

Explanation

- If $\mathbf{F} = \nabla V$, then the chain rule implies

$$\frac{d}{dt}\,\mathbf{V}(\mathbf{r}(t)) = \nabla V(\mathbf{r}(r)) \cdot \mathbf{r}'(t)$$

- Therefore, if $\mathbf{r}$ is a parametrization of C, then

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_C^b \nabla V(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_a^b \frac{d}{dt}\,\mathbf{V}(\mathbf{r}(t))\,dt \\
&= V(\mathbf{r}(b)) - V(\mathbf{r}(a))
\end{aligned}$$

Conclusion

- For a gradient vector field, the work done depends only on the endpoints of the path
- The total work can be computed by evaluating the potential function at the endpoints
- Lemma: If $\mathbf{F}$ is a gradient vector field and C is a simple closed differentiable path, then

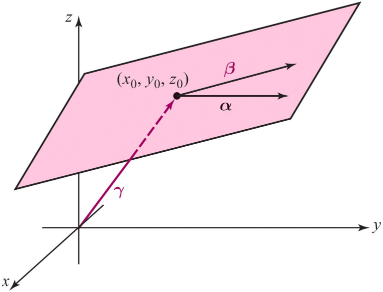
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Parametrized surfaces

- Example: A plane in $\mathbb{R}^3$ can be described as

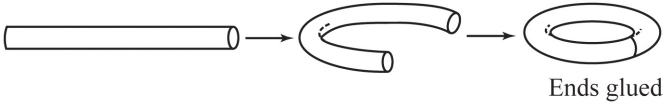
$$P(u,v) = \mathbf{p}_0 + u\mathbf{a} + v\mathbf{b}$$

where $\mathbf{p}_0$ is the "base point" and $\mathbf{a}$ and $\mathbf{b}$ are non-collinear vectors



Surfaces require two variables

- The parameters u and v represent the "latitude" and "longitude" of points on the surface
- Example: The torus



- Each point on the torus can be identified by $(\theta, \phi) \in [0, 2\pi)$

Orientability

- A surface is oriented if it has an umambiguous “top” and “bottom” or “inside” and “outside”
- A Möbius band is not oriented

Differentiability

- The parametrization function has the form

$$\Phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$

- $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$
- Define the tangent vectors

$$\begin{aligned} \mathbf{T}_u &= \partial\Phi/\partial u \\ \mathbf{T}_v &= \partial\Phi/\partial v \end{aligned}$$

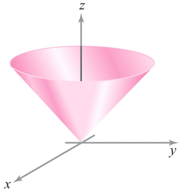
- If $\mathbf{T}_u \times \mathbf{T}_v \neq \mathbf{0}$ at a given (u_0, v_0), then we say that Φ is smooth there

Example (#2, p. 455): a cone

- $\Phi(u,v) = (u\cos v, u\sin v, u)$
- The tangent vectors are

$$\begin{aligned}\mathbf{T}_u &= (\cos v, \sin v, 1)\\ \mathbf{T}_v &= (-u\sin v, u\cos v, 0)\\ \mathbf{T}_u \times \mathbf{T}_v &= (-v\cos v, -u\sin v, u + (v-u)\cos^2 v)\end{aligned}$$

- At $\Phi(0,0) = (0,0,0)$, $\mathbf{T}_u \times \mathbf{T}_v = \mathbf{0}$



Tangent planes to a surface

- If $\mathbf{n}$ is a normal vector to a plane P at (x_0, y_0, z_0), then

$$\mathbf{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$$

- Consider the cone: $\Phi(u, v) = (u\cos v, u\sin v, u)$
- $\Phi(1, \pi) = (-1, 0, 1)$
- $\mathbf{T}_u \times \mathbf{T}_v(1, \pi) = (-1, 0, 1) = (\pi, 0, \pi)$
- A tangent plane is $(\pi, 0, \pi) \cdot (x + 1, 0, z - 1) = 0$ or $x + z = 0$.