Line Integrals

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Reading for this week

- Rest of Chapter 4
- Sections 7.1–7.3
- Next exam postponed until Friday, April 10
- CSUMS talks on Friday

Advertisement

- The Computational Science Training Program for Undergraduates in the Mathematical Sciences (CSUMS) is funded by the National Science Foundation
- Enrolls 11–13 students eaach year, starting August
- Pizza seminar on Mondays during the academic year
- Intensive 8-week summer reseearch project
- Honors credit available
- Stipend of up to $9,000 for one year
- Travel to a research conference

CSUMS qualifications

- U.S. citizen or permanent resident
- Math major or double major
- Application deadline: April 15
- Preferred: MAT 272, MAT 274/5, MAT 342/3 completed by December '09
- Preferred: CSE 100/110, 205 completed by Dec. '09
- Preferred: Expected graduation in May 2011 or later
- Friday speakers: Mary Cameron & Taylor Hines (cancer modeling)
- math.asu.edu/CSUMS

Discussion questions

1. (#9, p. 294) Sketch the vector field $\mathbf{F} = (y, -x)$ and a few flow lines.
2. (#8, p. 311) Sketch a few flow lines for $\mathbf{F} = (-3x, -y)$. Calculate $\operatorname{div} \mathbf{F}$ and explain why your answer is consistent with your sketch.
3. (#13, p. 311) Compute $\operatorname{curl} \mathbf{F}$ where $\mathbf{F} = (x, y, z)$

- Suppose that the density (mass per unit length) of a wire is $\mu(\mathbf{x})$
- If $\mathbf{r}(t)$, $a \le t \le b$, is the location of a segment of wire, then the total mass is

$$\int_C \mu(\mathbf{x})\, ds = \int_a^b \mu(\mathbf{r}(t)) \|\mathbf{r}'(t)\|\, dt$$

- Example: Suppose $\mu(x,y) = y$. Compute

$$\int_C y\, ds$$

where C is the upper half of the circle of radius 2.

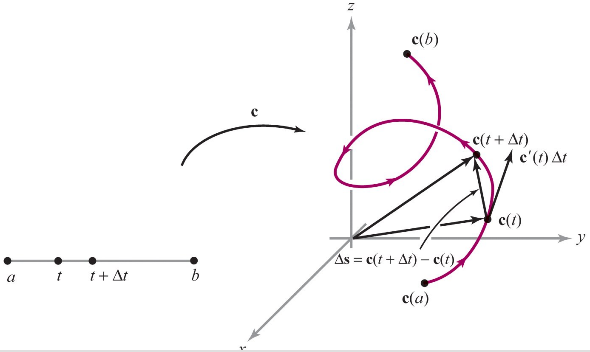
Solution

1. Parametrize the curve: $\mathbf{r}(t) = (2\cos t, 2\sin t)$, $0 \leq t \leq \pi$
2. Substitute in the integrand: $y = 2\sin t$
3. Compute ds: $ds = \|\mathbf{r}'(t)\|\, dt = 2\, dt$
4. Integrate:

$$\int_C y\, ds = \int_0^{\pi} 2 \cdot 2\sin t\, dt = 8$$

The line integral

- The work done by a constant force $\mathbf{F}$ over a displacement $\mathbf{d}$ is $W = \mathbf{F} \cdot \mathbf{d}$.
- Suppose $\mathbf{F}$ varies with position and we operate over a curve $\mathbf{c}(t)$



The line integral, continued

- Over the time interval $\Delta t = t_{i+1} - t_i$, the displacement is Δs
- The corresponding work is approximately

$$\Delta W \approx \mathbf{F}(\mathbf{c}(t_i)) \cdot \Delta s \approx \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\ \Delta t$$

so that

$$W \approx \sum_{i=1}^{n} \mathbf{F}(\mathbf{c}(t_i)) \cdot \mathbf{c}'(t_i)\ \Delta t$$

- Definition:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\ dt.$$

Example

- What is the work done by the vector field $\mathbf{F} = (y, -x)$ done on a particle that travels along the upper half of the circle of radius 2?
- Parametrize as $\mathbf{c}(t) = (2\cos t, 2\sin t)$, so

$$\begin{aligned} W &= \int_C (y, -x) \cdot d\mathbf{r} \\ &= \int_0^\pi (2\sin t, -2\cos t) \cdot (-2\sin t, 2\cos t)\, dt \\ &= -4\pi. \end{aligned}$$

The reparametrization theorem

- If $\mathbf{c}_1(t)$ $(a_1 \leq t \leq b_1)$ and $\mathbf{c}_2(t)$ $(a_2 \leq t \leq b_2)$ describe the same curve, then

$$\int_{a_1}^{b_1} \mathbf{F}(\mathbf{c}_1(t)) \cdot \mathbf{c}_1'(t)\, dt = \int_{a_2}^{b_2} \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt$$

  provided that $\mathbf{c}_2$ is orientation preserving, i.e., starts at the same place: $\mathbf{c}_1(a_1) = \mathbf{c}_2(a_2)$ and $\mathbf{c}_1(b_1) = \mathbf{c}_2(b_2)$

- An orientation reversing parametrization gives

$$\int_{a_1}^{b_1} \mathbf{F}(\mathbf{c}_1(t)) \cdot \mathbf{c}_1'(t)\, dt = -\int_{a_2}^{b_2} \mathbf{F}(\mathbf{c}_2(t)) \cdot \mathbf{c}_2'(t)\, dt$$

Equivalent notations

- Let $\mathbf{F} = (P, Q, R)$ and $\mathbf{c}(t)$ be a parametrization of the curve C for $a \le t \le b$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_C P\, dx + Q\, dy + R\, dz$$

- The rightmost integral is in differential form

Discussion question

1. Sketch the vector field $\mathbf{F} = (y, -x)$.
2. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the upper half of the circle of radius r centered around the origin, traversed counterclockwise.
3. Why is the work negative?
4. (#2a, p. 447) Let C be the unit circle. Evaluate

$$\int_C -y\,dx + x\,dy$$

5. As in #4, but C is the directed line segment from $(0,0)$ to $(1,1)$.