More on Vector Fields

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Reading for this week

- Rest of Chapter 4
- Sections 7.1–7.3
- Next exam postponed until Friday, April 10

Circular orbits

- Newton's second law: $\mathbf{F} = m\mathbf{a}$
- Consider a circular orbit of radius ρ:

$$\mathbf{r}(t) = \rho\left(\cos\frac{st}{\rho}, \sin\frac{st}{\rho}\right)$$

- The speed of motion is

$$\|\mathbf{r}'(t)\| = \left\|\rho\left(-\frac{s}{\rho}\sin\frac{st}{\rho}, \frac{s}{\rho}\cos\frac{st}{\rho}\right)\right\| = s.$$

Circular orbits, continued

- The acceleration is

$$\mathbf{r}''(t) = \rho\left(-\frac{s^2}{\rho^2}\cos\frac{st}{\rho}, -\frac{s^2}{\rho^2}\sin\frac{st}{\rho}\right) = -\frac{s^2}{\rho^2}\mathbf{r}(t).$$

- Note that $\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0$
- The direction of force is toward the center
- This centripetal force keeps the particle from flying off

Newton's gravitational law

- Consider a mass M at the origin and m at $\mathbf{r} = (x, y, z)$
- The gravitational potential is

$$V(\mathbf{r}) = \frac{GMm}{\rho} \quad \text{where} \quad \rho = \|\mathbf{r}\|$$

- If m moves in a circular orbit of radius ρ about M, then the gravitational force of M on m is

$$m\mathbf{r}''(t) = -\nabla V(\mathbf{r}) = -\frac{GMm}{\rho^3}\mathbf{r}(t)$$

Newton's gravitational law, continued

- Previous analysis shows that for a circular orbit of radius ρ and speed s,

$$m\mathbf{r}''(t) = -\frac{s^2 m}{\rho^2}\mathbf{r}(t) = -\frac{GMm}{\rho^3}\mathbf{r}(t)$$

which implies

$$\frac{s^2}{\rho^2}m = \frac{GMm}{\rho^3}, \quad \text{so} \quad s^2 = \frac{GM}{\rho}$$

- Planets in the Solar System have circular orbits, to a reasonable approximation
- Pluto is an exception (but isn't a planet!)

Kepler's Law

- Suppose the period of the planet is T
- Then $s = 2\pi\rho/T$
- Hence

$$s^2 = \frac{4\pi^2\rho^2}{T^2} = \frac{GM}{\rho},$$

- Kepler's Law:

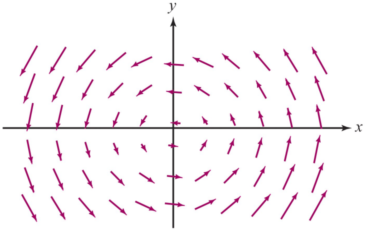
$$T^2 = \frac{4\pi^2\rho^3}{GM}$$

The square of the planet's period is proportional to the cube of its distance from the Sun.

Vector field

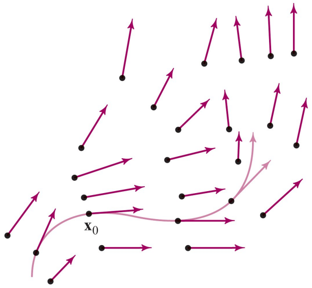
- A vector field is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ (usually $n = 2$ or 3)
- Leads to the differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}(t))$$



Differential equations

- The initial condition $\mathbf{x}_0$ is the starting point, say at $t = 0$
- A solution is a function $\mathbf{x}(t)$ such that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}'(t) = V(\mathbf{x}(t))$

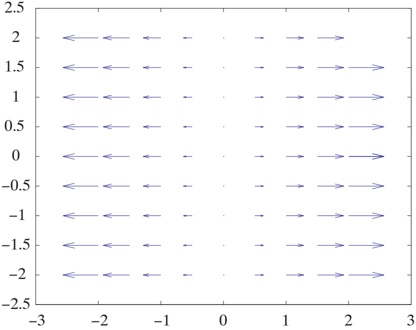


The flow of a differential equation

- The solution of $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ can be regarded as a function $\mathbf{x}(t) = \phi(\mathbf{x}_0, t)$
- ϕ is the flow associated with the differential equation
- Note: $\partial\phi/\partial t = \mathbf{F}(\phi(\mathbf{x}_0, t))$
- Makes explicit the dependence on the initial conditions
- Basic question: What happens to a typical set of initial conditions in a vector field?

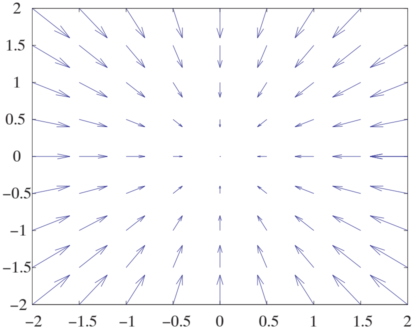
Example 1

- Consider $\mathbf{F}(x, y) = (x, 0)$
- What happens to a small box of particles around the origin?



Example 2

- Consider $\mathbf{F}(x, y) = (-x, -y)$
- What happens to a small box of particles around the origin?



Divergence

- In fluid dynamics, $\operatorname{div}\mathbf{F}$ is the rate of expansion per unit volume under the flow
- Define the operator

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

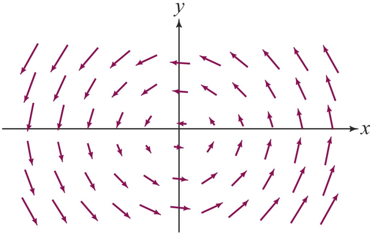
Then

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

- Take the first two components only in $\mathbb{R}^2$

Examples

- Given $\mathbf{F} = (x, 0)$, then $\operatorname{div} \mathbf{F} = 1$
- Fluid volume expands with time
- Given $\mathbf{F} = (-x, -y)$, then $\operatorname{div} \mathbf{F} = -2$
- Fluid volume contracts with time
- Given $\mathbf{F} = (-y, x)$, then $\operatorname{div} \mathbf{F} = 0$



The curl

- Given $\mathbf{F} = (P, Q, R)$, the curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$
$$= (R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k}$$

- In a fluid flow, $\operatorname{curl} \mathbf{F} = \mathbf{0}$ means there are no whirlpools at that point. (An infinitesimal paddlewheel does not rotate.)

Example (#7, p. 300)

- Let $\mathbf{F}(x,y,z) = (x, xy, 1)$
- Then

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & xy & 1 \end{vmatrix} \\ &= (0-0)\mathbf{i} - (0-0)\mathbf{j} + (y-0)\mathbf{k} \end{aligned}$$

A gradient vector field is curl free

- Theorem: $\nabla \times (\nabla V) = \mathbf{0}$.
- Proof:

$$\begin{aligned} \nabla \times (\nabla V) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} \\ &= (V_{zy} - V_{yz})\mathbf{i} - (V_{zx} - V_{xz})\mathbf{j} + (V_{yx} - V_{xy})\mathbf{k} \\ &= \mathbf{0}. \end{aligned}$$

- Example: A gravitational field has no curl.

Curls are divergence free

- Theorem: $\operatorname{div}\operatorname{curl}\mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$.
- Divergence free means there are no sources or sinks
- Curl free means there are no whirlpools (no rotations)

Discussion questions

1. (#9, p. 294) Sketch the vector field $\mathbf{F} = (y, -x)$ and a few flow lines.
2. (#8, p. 311) Sketch a few flow lines for $\mathbf{F} = (-3x, -y)$. Calculate $\operatorname{div}\mathbf{F}$ and explain why your answer is consistent with your sketch.
3. (#13, p. 311) Compute $\operatorname{curl}\mathbf{F}$ where $\mathbf{F} = (x, y, z)$