Improper integrals

Eric Kostelich

ARIZONA STATE UNIVERSITY
DEPT. OF MATHEMATICS AND STATISTICS

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Announcements

- Reading for this week: Section 6.4 and Chapter 4
- Goals for the rest of the course: Discuss the mathematics underlying Maxwell's equations

Review: Improper integrals in one dimension

- Example: Evaluate $\int_1^\infty \frac{dx}{x^2}$.
- Treat the integral as a limit:

$$\lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \lim_{b\to\infty}\left(-\frac{1}{x}\Big|_1^b\right) = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1.$$

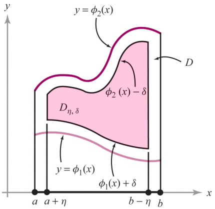
- We say that the improper integral exists and equals L if the corresponding limit exists and equals L.

The idea is similar for multiple integrals

- If the integrand f becomes undefined at one or more points in a domain D, we "shrink" D slightly to $D_{\eta,\delta}$
- We say that $\iint_D f\,dA$ exists and equals L if

$$\lim_{(\eta,\delta)\to(0,0)} \iint_{D_{\eta,\delta}} f\,dA$$

exists and equals L.



Example (p. 408)

- Evaluate
$$\iint_{[0,1]\times[0,1]} \frac{1}{\sqrt[3]{xy}}\, dA$$
over the unit square.
- Difficulty: $(xy)^{-1/3}$ is undefined at the origin.
- Consider (if it exists)
$$\lim_{(\eta,\delta)} \int_{\eta}^{1-\eta} \int_{\delta}^{1-\delta} \frac{1}{\sqrt[3]{xy}}\, dy\, dx$$

Example, continued

- This gives

$$\lim_{(\eta,\delta)\to(0,0)} \frac{3}{2}\left((1-\eta)^{2/3}-\eta^{2/3}\right)$$

$$\times\frac{3}{2}\left((1-\delta)^{2/3}-\delta^{2/3}\right)=1.$$

Most such limits aren't so easy

- Example:
$$\iint_D \frac{1}{\sqrt{1-x^2-y^2}}\,dA$$
where D is the unit disk.
- This integral gives the area of the upper hemisphere of the unit sphere.
- Individual limits in x and y are hard to evaluate

Fubini's Theorem for improper integrals

- Suppose $f(x,y) \geq 0$ on D. If any one of the integrals

$$\iint_D f(x,y)\,dA, \qquad \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

$$\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$

exists, then so do all the others, and they are all equal.

Consider polar coordinates

- Except on the boundary of the disk D, we have

$$\iint_D \frac{1}{\sqrt{1-x^2-y^2}}\,dA = \iint_{P(D)} \frac{r}{\sqrt{1-r^2}}\,dr\,d\theta$$

so evaluate

$$\lim_{\delta \searrow 0} \int_0^{1-\delta} \int_0^{2\pi} \frac{r}{\sqrt{1-r^2}}\,d\theta\,dr$$

- This gives

$$\lim_{\delta \searrow 0} 2\pi \int_0^{1-\delta} \frac{r}{\sqrt{1-r^2}}\,dr$$

Polar coordinates, continued

Therefore,

$$2\pi \lim_{\delta \searrow 0} -\sqrt{1-r^2}\Big|_0^{1-\delta} = 2\pi \lim_{\delta \searrow 0} -\sqrt{1-(1-\delta)^2}+1$$
$$= 2\pi.$$

The surface area of the unit sphere is 4π.