Lagrange Multipliers and Introduction to Multiple Integration

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Announcements

- Reading for this week: Sections 5.1–5.5

Review: Basic geometry of Lagrange multipliers

- If a level curve of f achieves an extreme value on the constraint curve $g = \text{constant}$ at the point $\mathbf{x}_0$, then $\nabla f(\mathbf{x}_0)$ and $\nabla g(\mathbf{x}_0)$ are parallel
- In other words, $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$
- The scalar λ is called a Lagrange multiplier
- Important: If $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$, then $\mathbf{x}_0$ is only a candidate point for an extreme value of f
- Verification may be empirical or may required complicated second-derivative tests (bordered Hessians)

Example 1: the extreme values of $f(x,y) = x^2 + xy + y^2$ on the unit circle

- We require $\nabla f(x,y) = (2x+y, x+2y) = \lambda \nabla g(x,y) = \lambda(2x, 2y)$
- Hence

$$\begin{aligned} 2x + y &= 2\lambda x \\ x + 2y &= 2\lambda y \\ x^2 + y^2 &= 1 \end{aligned}$$

- Multiply the first equation by x and the second by y:

$$\begin{array}{rcl} 2(1-\lambda)x^2 + xy &=& 0 \\ 2(1-\lambda)y^2 + xy &=& 0 \\ \hline 2(1-\lambda)(x^2 - y^2) &=& 0 \end{array}$$

Example 1, continued

- Either $\lambda = 1$ or $x^2 = y^2$
- If $\lambda = 1$ then
  - either $x = 0$ (from the first equation) so $y = \pm 1$
  - or $y = 0$ (from the second equation) so $x = \pm 1$
- Hence four candidate points for extrema are $(0, \pm 1)$ and $(\pm 1, 0)$

Example 1, continued

- If $x^2 = y^2$ then the constraint equation implies $2x^2 = 1$, so $x = \pm 1/\sqrt{2}$
- This gives four more points to evaluate:

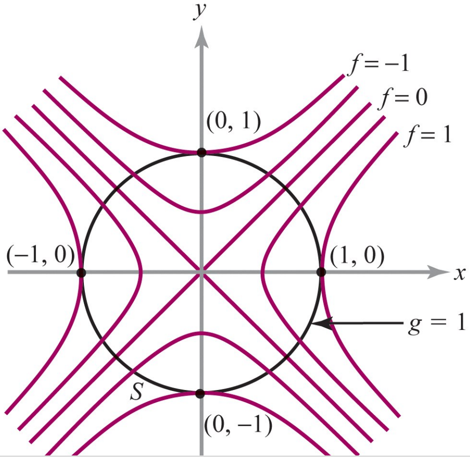
$$\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}} \right)$$

- Evaluate $f(x,y) = x^2 + xy + y^2$ at each candidate point:

$$\begin{aligned} f(0,\pm 1) &= -1, \quad f(\pm 1, 0) = 1 \\ f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) &= f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2} \\ f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{3}{2} \end{aligned}$$

Example 2 (pp. 229–230)

- Find the extreme values of $f(x, y) = x^2 - y^2$ on the unit circle.



Example 2, continued: $f(x,y) = x^2 - y^2$ on unit circle

- We must solve $\nabla f(x,y) = \lambda \nabla g(x,y)$
- Here $\nabla f(x,y) = (2x, -2y) = \lambda(2x, 2y)$
- This gives the equations

$$\begin{aligned} 2x &= 2\lambda x \\ -2y &= 2\lambda y \\ x^2 + y^2 &= 1 \end{aligned}$$

- The first equation implies that either $\lambda = 1$ or $x = 0$.
- If $x = 0$ then $y = \pm 1$ so $\lambda = -1$.
- If $\lambda = 1$ then $y = 0$ and $x = \pm 1$.

Example 2, continued

- This gives four points to be evaluated for potential extreme values:

$$(\pm 1, 0) \quad \text{and} \quad (0, \pm 1).$$

- We have $f(\pm 1, 0) = 1$ and $f(0, \pm 1) = -1$
- That these are extreme values can be verified by inspection

Multiple constraints

- There are as many Lagrange multipliers as there are constraints
- Example: Find the maximum values of $f = x + y + z$ subject to $g_1 = x^2 + y^2 = 2$ and $g_2 = x + z = 1$
- We must solve the system given by the constraints and

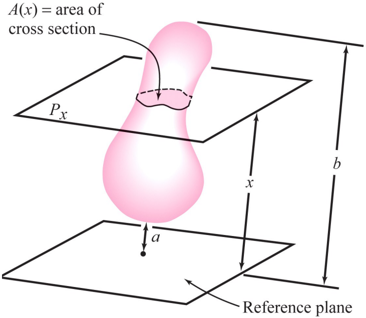
$$\begin{aligned} \nabla f &= \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ (1,1,1) &= \lambda_1 (2x, 2y, 0) + \lambda_2 (1,0,1) \end{aligned}$$

- The remaining details are in Example 5 on p. 233

Cavalieri's principle

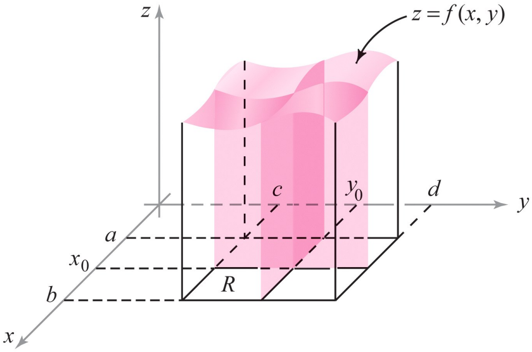
- If $A(x)$ is the area of a cross-section at height x, then the volume is

$$\int_a^b A(x)\,dx$$



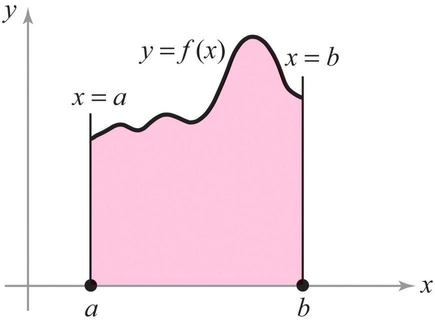
Example: find the volume under a surface

- Suppose $f : R = [a,b] \times [c,d] \to \mathbb{R}$
- Use Cavalieri's principle to evaluate $V = \iint_R f(x,y)\, dA$



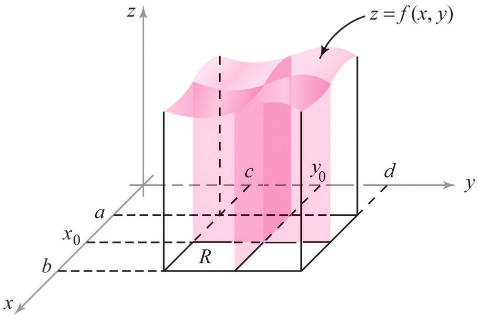
The area under a curve

- If $f : [a,b] \to \mathbb{R}$ is continuous and nonnegative, then $\int_a^b f(x)\,dx$ is the area under the graph of $f(x)$



Use the same idea for the area of a slice

- Fix $y = y_0$. Then $\int_a^b f(x, y_0)\, dx$ is the area $A(y_0)$ of the region at y_0 under the graph of f
- In other words, slice from “North to South”



Cavalieri's principle yields the volume

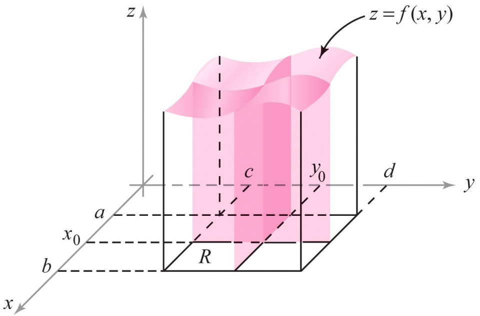
- Given $A(y_0)$ for $c \le y_0 \le d$, the volume is

$$V = \int_c^d A(y)\,dy = \int_c^d \underbrace{\left[\int_a^b f(x,y)\,dx\right]}_{\text{fixed } y} dy$$

- This is called an iterated integral

We can do the same thing in the other direction

- Fix $x = x_0$. Then $\int_c^d f(x_0, y)\, dy$ is the area $A(x_0)$ of the region at x_0 under the graph of f
- In order words, slice from “West to East”



Cavalieri's principle again yields the volume

- Given $A(x_0)$ for $a \leq x_0 \leq b$, the volume is

$$V = \int_a^b A(x)\,dx = \int_a^b \underbrace{\left[\int_c^d f(x,y)\,dy\right]}_{\text{fixed } x} dx$$

- Fubini's theorem guarantees that if f is continuous on $[a,b] \times [c,d]$, the order of integration doesn't matter:

$$V = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

Example: Problem 1a, p. 325

- Let $R = [-1,1] \times [0,1]$. Evaluate $\iint_R (x^4y + y^2)\, dA$.
- Method 1: Compute

$$
\begin{aligned}
\int_{-1}^{1} \left[\int_0^1 (x^4y + y^2)\, dy \right] dx &= \int_{-1}^{1} \left[\tfrac{1}{2}x^4y^2 + \tfrac{1}{3}y^3 \Big|_0^1 \, dx \right. \\
&= \int_{-1}^{1} \left(\tfrac{1}{2}x^4 + \tfrac{1}{3} \right) dx \\
&= \tfrac{1}{10}x^5 + \tfrac{1}{3}x \Big|_{-1}^{1} \\
&= \tfrac{13}{15}.
\end{aligned}
$$

Fubini's theorem lets us proceed the other way

- Method 2: Compute

$$\begin{aligned}\int_0^1 \left[\int_{-1}^1 (x^4 y + y^2)\,dx\right] dy &= \int_0^1 \left[\tfrac{1}{5}x^5 y + y^2 x\Big|_{-1}^1\right. dy \\ &= \int_0^1 \left(\tfrac{2}{5}y + 2y^2\right) dy \\ &= \tfrac{1}{5}y^2 + \tfrac{2}{3}y^3\Big|_0^1 \\ &= \frac{13}{15}.\end{aligned}$$

Useful strategy

- Fubini's theorem lets you evaluate a double integral in whichever order is easiest
- The same idea extends to triple integrals