Lagrange Multipliers

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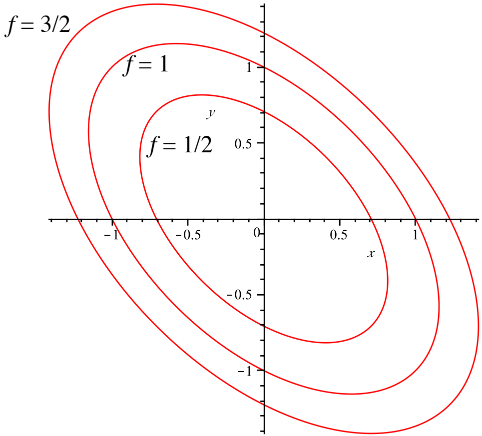
Feb. 13, 2009

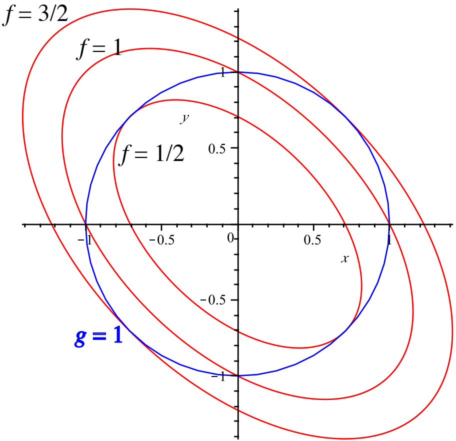
Announcements

- Reading for this week: Sections 2.6, 3.1, 3.3, 3.4

Motivation: Level curves for $f(x, y) = x^2 + x + y^2$



What are the maxima of f on the unit circle?



Basic geometry

- If a level curve of f achieves a minimum value on the constraint curve $g = \text{constant}$ at the point $\mathbf{x}_0$, then the two curves must be tangent at $\mathbf{x}_0$
- This implies that $\nabla f(\mathbf{x}_0)$ and $\nabla g(\mathbf{x}_0)$ are parallel
- In other words, $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$ when f assumes an extreme value at $\mathbf{x}_0$
- The scalar λ is called a Lagrange multiplier

Caveat

- The Lagrange multiplier method requires that we find a scalar λ and a point $\mathbf{x}_0$ at which $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$
- The point $\mathbf{x}_0$ found in this way is only a candidate point for an extreme value of f!
- Whether $f(\mathbf{x}_0)$ really is an extreme value must be assessed by other means

Example

- Find the extreme values of $f(x,y) = x^2 + xy + y^2$ on the unit circle.
- Constraint: $g(x,y) = x^2 + y^2 = 1$
- We require
  $\nabla f(x,y) = (2x + y, x + 2y) = \lambda \nabla g(x,y) = \lambda(2x, 2y)$
- This leads to the following nonlinear system of equations in the unknowns x, y, and λ:

$$\begin{aligned} 2x + y &= 2\lambda x \\ x + 2y &= 2\lambda y \\ x^2 + y^2 &= 1 \end{aligned}$$

The solutions methods are *ad hoc*

- There is no general formula for solving such systems of equations
- Any algebraic solution is a matter of trial and error
- Large systems are solved numerically

Back to $f(x,y) = x^2 + x + y^2$ on the unit circle

- We have

$$\begin{aligned} 2x + y &= 2\lambda x \\ x + 2y &= 2\lambda y \\ x^2 + y^2 &= 1 \end{aligned}$$

- Multiply the first equation by x^2 and the second by y^2:

$$\begin{aligned} 2(1-\lambda)x^2 + xy &= 0 \\ 2(1-\lambda)y^2 + xy &= 0 \\ \hline 2(1-\lambda)(x^2 - y^2) &= 0 \end{aligned}$$

- This implies $x = \pm y$ on the unit circle

Lagrange multiplier solution, continued

- Hence $x^2 + y^2 = 2x^2 = 1$, so $x = \pm 1/\sqrt{2}$
- There are four pairs of points to evaluate:

$$\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$$

- Evaluate $f(x,y) = x^2 + xy + y^2$ at each point to find the extrema:

$$\begin{aligned} f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) &= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} = \frac{1}{2} \\ f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2} \end{aligned}$$

and similarly for the other two points.