Extreme values of functions of two variables

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Announcements

- Reading for this week: Sections 2.6, 3.1, 3.3, 3.4

Review: The second derivative test for $f(x, y)$

- Find all critical points: $\nabla f(x_0, y_0) = (0, 0)$
- If $D = f_{xx} f_{yy} - (f_{xy})^2 = 0$ then the test fails
- If $D < 0$ then (x_0, y_0) is a saddle
- If $D > 0$ then $f(x_0, y_0)$ is a minimum if $f_{xx} > 0$ and $f(x_0, y_0)$ is a maximum if $f_{xx} < 0$
- The extreme values are isolated: if $f(x_0, y_0)$ is a minimum, then $f(x, y) > f(x_0, y_0)$ for all points sufficiently close to (x_0, y_0) (analogously for maxima)

This process isn't easy

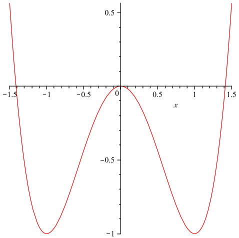
- Finding extreme values is more complicated in $\mathbb{R}^2$ than $\mathbb{R}$
- Solving for $\nabla f(x_0, y_0) = (0, 0)$ may be difficult or impossible to do analytically
- Formulating a set of criteria that works for all f is tricky
- No general rules for functions of many variables

Optimization is an active area of research

- Example: the Traveling Salesman Problem
- Find the shortest route that allows a salesman to visit each city in a list exactly once
- How should a UPS truck be routed to make its deliveries?
- Many subproblems: What is the best way to pack the parcels in each truck?

Local vs. global maxima

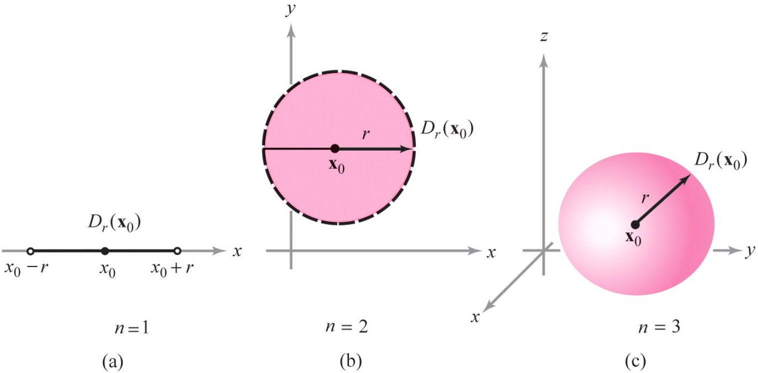
- Example: consider $f(x) = x^4 - 2x^2$
- Then $f'(x) = 4x^3 - 4x = 4(x^3 - x)$
- The critical points are -1, 0, and 1
- The points $x = \pm 1$ are global minima
- The point $x = 0$ is a local (relative) maximum



Now consider a bounded domain

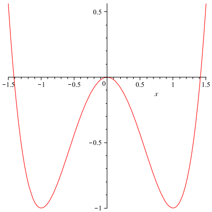
- Example: The function $f(x) = x^4 - 2x^2$ has both maximum and minimum values on $[-2, 2]$
- We consider the critical points and the boundary points of the domain
- The same idea applies in $\mathbb{R}^2$

Open disks


$D_r(\mathbf{x}_0)$
$x_0 - r$
x_0
$x_0 + r$
x
$n = 1$
(a)
y
r
$\mathbf{x}_0$
$D_r(\mathbf{x}_0)$
x
$n = 2$
(b)
z
$D_r(\mathbf{x}_0)$
r
$\mathbf{x}_0$
y
x
$n = 3$
(c)

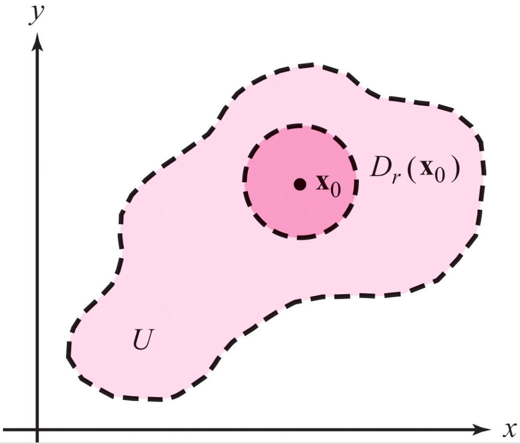
Remarks

- The open interval $(-2,2) = \{x : -2 < x < 2\}$ has no largest or smallest value
- The function $f(x) = x^4 - 2x^2$ attains no largest value on $(-2,2)$
- But it does have a largest value on $[-2,2] = \{x : -2 \le x \le 2\}$, namely $f(\pm 2) = 8$



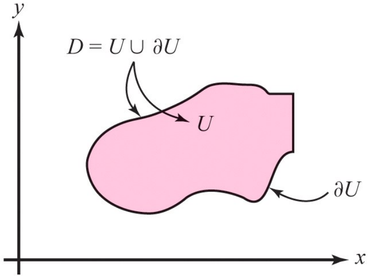
Open sets

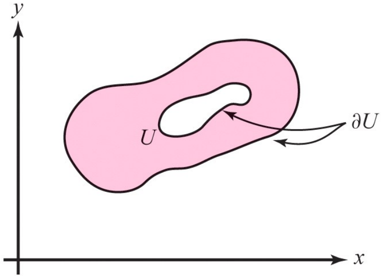
- The set U is open if for every $\mathbf{x} \in U$, there exists $r > 0$ such that $D_r(\mathbf{x}) \subset U$



Boundary points

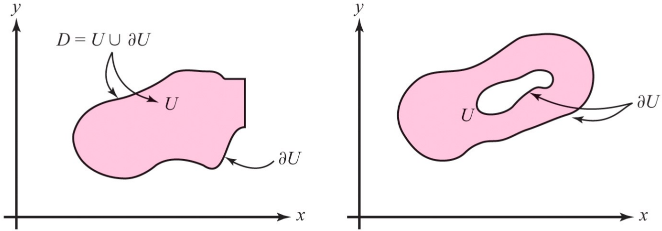
- The point $\mathbf{b}$ is a boundary point of U if for every $r > 0$, the disk $D_r(\mathbf{b})$ contains points in U and points in U^c
- ∂U is the set of boundary points of U
- The closure of U is the set $\overline{U} = U \cup \partial U$





Bounded and closed sets

- The set U is bounded if there exists $M > 0$ such that $U \subset D_M(\mathbf{0}$
- The set U is closed if U contains all of its boundary points, i.e., $\overline{U} = U$.
- Example: The sets below are both bounded and closed



Extrema of functions on closed and bounded sets

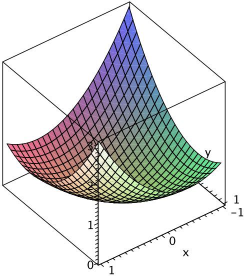
- Let U be a closed bounded set
- Look for critical points of f in U and evaluate them as before
- Then find the extreme values of f on ∂U
- Theorem: If U is closed and bounded, then $f : U \to \mathbb{R}$ attains maximum and minimum values on U.

Example 1

- Consider $U =$ closed unit disk and $f(x,y) = x^2 + y^2$
- The only critical point of f is $(0,0)$ (a minimum)
- Next evaluate f on $\partial U =$ unit circle
- The maximum value of f is 1 (attained on ∂U)
- For more complicated functions, find a parametrization $\mathbf{c}(t)$ of ∂U and examine the critical points of $f(\mathbf{c}(t))$

Example 2

- Consider $U =$ closed unit disk and $f(x,y) = x^2 + xy + y^2$
- $\nabla f = (2x+y, x+2y)$ so $(0,0)$ is the only critical point
- $D(0,0) = 2 \times 2 - 1 = 3$ and $f_{xx}(0,0) = 2 > 0$
- Hence $(0,0)$ is a relative minimum



Example 2, continued

- The other extreme values of f must occur on the boundary
- Let $\mathbf{c}(t) = (\cos t, \sin t)$
- Then $\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = 2\cos^2 t - 1$
- The critical points are $t = \pm\pi/4$
- $f(\mathbf{c}(\pi/4)) = 3/2$ and $f(\mathbf{c}(-\pi/4)) = 1/2$
- Hence the maximum value of f over the closed unit disk is $3/2$ and the minimum value is 0

Extreme values of functions subject to constraints

- An active research area for functions of many variables
- In the previous example, we considered the extreme values of f subject to being on the unit circle C, that is, $f|_C$
- More generally, we can consider the extreme values of $f(\mathbf{x})$ subject to $g(\mathbf{x}) = \text{constant}$
- Example: Find the point of a surface S that is closest to a fixed point $\mathbf{p}$
- Minimize $f(\mathbf{x}) = \|\mathbf{x} - \mathbf{p}\|$ subject to $\mathbf{x} \in S$
- Can regard S as the level surface for $g(\mathbf{x}) = \text{constant}$

Motivation

- Consider a path $\mathbf{c}(t)$ on the level surface S of g
- Then $g(\mathbf{c}(t)) = \text{constant}$ for all t
- Thus $\frac{d}{dt}g(\mathbf{c}(t)) = \frac{d}{dt}\text{constant} = 0$
- The chain rule implies $\frac{d}{dt}g(\mathbf{c}(t)) = \nabla g(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = 0$
- Since $\mathbf{c}'(t)$ is tangent to S, ∇g is perpendicular to S
- In what follows, assume $\nabla g \neq \mathbf{0}$

Lagrange multipliers

- Now suppose $f|_S$ has a maximum at $\mathbf{c}(0) = \mathbf{x}_0$
- This requires $\frac{d}{dt}f(\mathbf{c}(t))|_{t=0} = \nabla f(\mathbf{c}(0)) \cdot \mathbf{c}'(0) = 0$
- Hence $\nabla f(\mathbf{c}(0)) = \nabla f(\mathbf{x}_0)$ is orthogonal to S at $\mathbf{x}_0$
- But so is $\nabla g(\mathbf{x}_0)$—hence the two gradient vectors are parallel
- That is, $\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$ if f reaches a maximum at $\mathbf{x}_0$
- λ is called a Lagrange multiplier