The Derivative

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Reading for this week

- Sections 2.2–2.4

Recap: Discussion questions from last time

Which of the following functions is continuous at $x = 0$? Try writing an epsilon-delta argument.

1. $f(x) = |x|$
2. $f(x) = 1/x$
3. $f(x) = \sqrt{x}$
4. $f(x) = x^2 + 1$

Example

- Consider $f(x) = x^2 + 1$
- Clearly defined at $x = 0$
- Need to show that $\lim_{x \to 0} f(x) = 1$
- Let $\varepsilon > 0$ be given. Need to find δ such that

$$|f(x+\delta) - f(x)| = |f(\delta) - 1| = \delta^2 < \varepsilon.$$

- It suffices to choose $0 < \delta < \sqrt{\varepsilon}$

Limits in $\mathbb{R}^2$

- There is no analog of l'Hôpital's rule in more than one dimension
- The easy case: the function in question is the sum, product, or quotient of continuous functions (and the denominator isn't zero)
- Example: (#10a, §2.2)

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x+1} = \frac{e^0}{1} = 1$$

Tricky limits

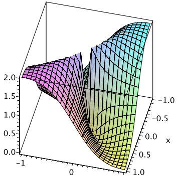
- If the limit has the form $0/0$, then there are *ad hoc* approaches
- One method: If you can show that the limiting value depends on the path, then the limit does not exist
- One useful choice: consider approaching the origin along the line $y = mx$ for various values of m

Example: #10c, §2.2

- $$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$
- Approach the origin along $y = mx$, which becomes

$$\lim_{x\to 0} \frac{(x-mx)^2}{x^2+(mx)^2} = \lim_{x\to 0} \frac{x^2(1-m)^2}{x^2(1+m^2)} = \frac{(1-m)^2}{1+m^2}$$

- Since the result depends on m, the original limit does not exist



Partial derivatives

- Suppose $f : \mathbb{R}^2 \to \mathbb{R}$, i.e., $z = f(x,y)$
- The partial derivative of f with respect to the variable x is the rate of change of f when y is held constant:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

whenever the limit exists.

- Similarly, we can hold x constant and let y vary:

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

The computation rules are as in 1-variable calculus

- Example: $f(x,y) = y\sin 2x$
- To compute $\partial f/\partial x$, treat y as constant:

$$\frac{\partial f}{\partial x} = 2y\cos 2x$$

- To compute $\partial f/\partial y$, treat x as constant:

$$\frac{\partial f}{\partial y} = \sin 2x$$

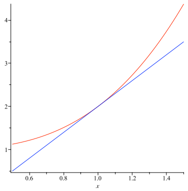
- The notation $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ indicates that the derivative is evaluated at the point (x_0,y_0)

The tangent line

- In 1-variable calculus, the tangent line approximation to the function $f(x)$ at the point $x = x_0$ is

$$L(x) = f(x_0) + f'(x_0) \cdot (x - x_0)$$

- Example: $f(x) = x^3 + 1$ at $x_0 = 1$
- $f(1) = 2$ and $f'(1) = 3$, so $L(x) = 2 + 3(x - 1) = 3x - 1$



The Gradient

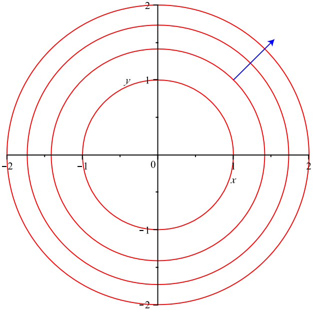
- The gradient of the function $f(x,y)$ is the vector

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

- Key geometric fact: $\nabla f(x_0, y_0)$ points in the direction of the fastest increase of f

Example

- Let $f(x,y) = x^2 + y^2$. Then $\nabla f = (2x, 2y)$
- The level curves of f are concentric circles
- $\nabla f(1,1) = (2,2)$



The tangent plane

- The tangent plane approximation of $f(\mathbf{x})$ at the point $\mathbf{x}_0$ is

$$P(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

- Note the parallel with the 1-dimensional case:

$$L(x) = f(x_0) + f'(x_0) \cdot (x - x_0)$$

- Example: $f(x,y) = x^2 + y^2$ at $\mathbf{x}_0 = (1,1)$:

$$P(x,y) = 2 + (2,2) \cdot (x-1, y-1) = 2x + 2y - 2$$

The tangent plane is useful for approximation

- Example: Given $f(x,y) = x^2 + y^2$. Estimate $f(1.1, 1.05)$.
- The tangent plane approximation is $P(\mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$
- Let $\mathbf{x}_0 = (1,1)$ be the base point:

$$P(x,y) = f(1,1) + \nabla f(1,1) \cdot (x-1, y-1) = 2x + 2y - 2$$

- This gives $P(1.1, 1.05) = 2.2 + 2.1 - 2 = 2.3$
- Note that $f(1.1, 1.05) = 2.31$

Discussion question

- Estimate $\left(0.99e^{0.02}\right)^8$ by linearizing an appropriate function $f(x,y)$ about a suitable base point.
- Solution: Consider $f(x,y) = (xe^y)^8$ at $(x_0, y_0) = (1,0)$
- $f(1,0) = 1$ and
  $\nabla f(1,0) = \left(8x^7e^{8y}, 8x^8e^{8y}\right)\Big|_{(1,0)} = (8,8)$
- Use the tangent plane $P(x,y) = 1 + (8,8)\cdot(x-1,y)$ to approximate f near $(1,0)$:

$$f(0.99, 0.02) \approx 1 + (8,8)\cdot(-0.01, 0.02) = 1.08.$$

The “exact” value is $1.082851\ldots$