Continuity and limits

Eric Kostelich

ARIZONA STATE UNIVERSITY
DEPT. OF MATHEMATICS AND STATISTICS

Jan. 26, 2009

Reading for this week

- Sections 2.2–2.4

Recap: Discussion questions from last time

1. (#4, p. 73) Describe the surfaces $r = \text{constant}$, $\theta = \text{constant}$, and $z = \text{constant}$ in the cylindrical coordinate system
2. Likewise, $\rho = \text{constant}$ and $\phi = \text{constant}$ in spherical coordinates
3. (#6–7, p. 107) Sketch the level curves of the function $f(x,y) = \sqrt{x^2 + y^2}$ and $g(x,y) = x^2 + y^2$ for $c = 0, 1, 2, 3, 4, 5$

Some history

- Calculus was developed in the 1600s by Newton and extended in the 1700s by Euler
- Had spectacular success in predicting the motions of the planets
- But lacked a rigorous foundation until the 1800s
- Some very smart people struggled for a long time to find the “right” definitions
- Example: The Newton quotient:

$$\frac{f(x+h)-f(x)}{h}$$

Morris Kline, *The Loss of Certainty*

- The extrapolation to a limit did not have a firm logical foundation in the 1600s.
- Bishop George Berkeley discerned a threat to religion and wrote a treatise in 1734 attacking the idea in general and Edmund Halley in particular:

  The Analyst Or a Discourse Addressed to an Infidel Mathematician. Wherein It Is Examined Whether the Object, Principles, and Inferences of the Modern Analysis Are More Distinctly Conceived, or More Evidently Deduced, than Religious Mysteries and Points of Faith.

Berkeley was no fool

- Newton's algebra used h during some steps, then dropped h in others because h was now zero.
- Berkeley objected that such reasoning was a "defiance of the law of contradiction" and would not be tolerated in theology.
- Of Newton's fluxions:

  > "They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities."

Berkeley's parting shot

"Whether mathematicians who are so delicate in religious points, are strictly scrupulous in their own science? Whether they do not submit to authority, take things upon trust, and believe points inconceivable? Whether they have not their mysteries, and what is more, their repugnances and contradictions?"

People took Berkeley's criticisms to heart

- Many people, especially Euler, tried to answer Berkeley's objections.
- Lagrange's objection:

> "The method has the great inconvenience of considering quantities in the state in which they cease, so to speak, to be quantities; for though we can always well conceive the ratios of two quantities, as long as they remain finite, that ratio offers to the mind no clear and precise idea, as soon as its terms both become nothing at the same time."

The contest

In 1784, Lagrange proposed that the Berlin Academy of Sciences award a prize for solving the problem of the infinite (and infinitesimal):

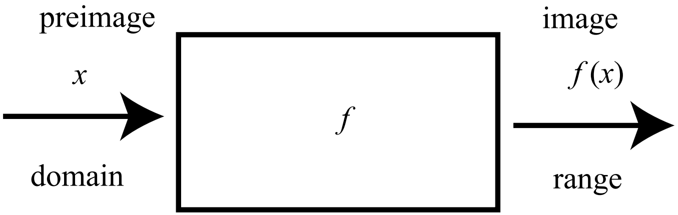
> "The Academy . . . desires an explanation of how it is that so many correct theorems have been deduced from a contradictory supposition, together with an enunciation of a sure, a clear, in short a truly mathematical principle that may properly be substituted for that of the *infinite* without, however, rendering investigations carried out by its means overly difficult or overly lengthy. It is required that the subject be treated in all possible generality and with all possible rigor, clarity, and simplicity."

Augustin-Louis Cauchy (1789–1857)

- Used the limit concept as the starting point of calculus
- He is generally credited with the epsilon-delta formulation of limit. (See *The American Mathematical Monthly* **90** (March 1983), 185–194, available online at www.maa.org/pubs/Calc_articles/ma002.pdf)
- But even Cauchy made many mistakes:
  - assumed that continuity implies differentiability
  - made false statements about infinite series

Notation and terminology

- Domain: The set D over which f is defined
- Range: The set $f(D)$, i.e., $\{y : y = f(x),\ x \in D\}$
- Notation: $f : D \to \mathbb{R}$ is a real-valued function i.e., $f(D) \subset \mathbb{R}$
- Codomain: The set into which f maps its domain



Examples

- $f(x) = 2x + 3$ has domain $\mathbb{R}$ and range $\mathbb{R}$
- The domain and range of $f(x) = \sqrt{x}$ are the nonnegative real numbers
- $f(x) = x^2$ has domain $\mathbb{R}$ and range $\mathbb{R}^+ \cup \{0\}$. The codomain is $\mathbb{R}$.
- $f(x) = \log x$ has domain $\mathbb{R}^+$ and range $\mathbb{R}$

Definition of the limit of a function

Definition

We say that the limit of f as x tends to a is L and write $\lim_{x \to a} f(x) = L$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

Key points to note

1. f does not have to be defined at a
2. You get to pick ε. The trick is to find a δ that works.
3. The limit must be the same whether a is approached from the left or the right

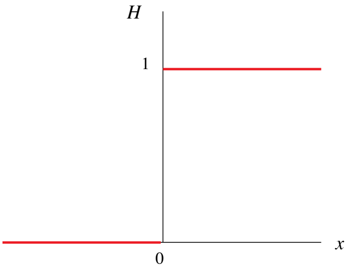
Example 1

- What is $\lim_{x \to 0}(2x+3)$?
- Suppose I pick $\varepsilon = \frac{1}{10}$. I need to find δ such that $|(2x+3)-3| < \frac{1}{10}$ whenever $|x-0| < \delta$.
- One answer: $\delta = \frac{1}{100}$. Any δ such that $0 < \delta < \frac{1}{20}$ will do.

Example 2

- The Heaviside function is defined as

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1 & x > 0 \end{cases}$$



Example 2, continued

- What is $\lim_{x \to 1/2} H(x)$?
- $\lim_{x \to 0} H(x)$?
- $\lim_{x \to 1/2} H(x) = 1$
- $\lim_{x \to 0} H(x)$ is undefined.

Example 3

- Let

$$f(x) = \begin{cases} 0, & x = 0 \\ 1 & x \neq 0 \end{cases}$$

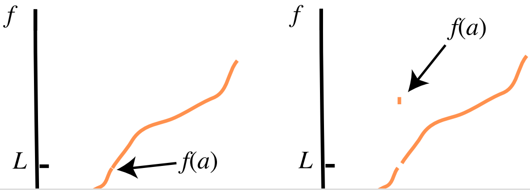
- What is $\lim_{x \to 0} f(x)$?
- $\lim_{x \to 0} f(x) = 1$

Continuity at a point

Definition

We say that f is continuous at a if

1. $\lim_{x \to a} f(x) = L$ and
2. $f(a) = L$
3. Note: a one-sided limit may be substituted if the function's domain requires



Example 1

- The function $f(x) = 2x + 3$ is continuous at every x.
- Proof: $f(x)$ is defined at every $x \in \mathbb{R}$.
- Given $\varepsilon > 0$, choose $\delta < \varepsilon/2$. Then

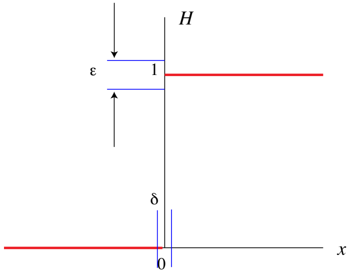
$$|f(x+\delta) - f(x)| = \big|[2(x+\delta) - 3] - [2x - 3]\big| = |2\delta| < \varepsilon.$$

Example 2

- The Heaviside function is defined as

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1 & x > 0 \end{cases}$$

- The Heaviside function is not continuous at $x = 0$



Basic theorems about continuous functions

- Suppose f and g are continuous functions on $\mathbb{R}$ and c is any constant.
- Then cf, $f+g$ and $f \times g$ are continuous
- f/g is continuous wherever $g \neq 0$
- $f \circ g$ is continuous

Discussion questions

Which of the following functions is continuous at $x = 0$? Try writing an epsilon-delta argument.

1. $f(x) = |x|$
2. $f(x) = 1/x$
3. $f(x) = \sqrt{x}$
4. $f(x) = x^2 + 1$