In-class exercises, April 28, 2009

1. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable. Prove that $\nabla \times \nabla f = \mathbf{0}$.

2. Recall that the *divergence* of $\mathbf{F} = (P, Q, R)$, written as $\operatorname{div} \mathbf{F}$ or $\nabla \cdot \mathbf{F}$, is defined as
$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$
Gauss's theorem (also called the divergence theorem) states that if S is a simple smooth closed surface (such as a sphere) enclosing a volume V, then
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{F})\, dV.$$
Suppose that S is the unit sphere and $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ both directly and by using Gauss's theorem.

3. (Problem 23, p. 549) Let $\mathbf{F} = x^2\mathbf{i} + (2xy + x)\mathbf{j} + z\mathbf{k}$. Let C be the circle $x^2 + y^2 = 1$ and S the disk $x^2 + y^2 \leq 1$ within the xy plane.
   (a) Find the flux of $\mathbf{F}$ out of S.
   (b) Determine the circulation of $\mathbf{F}$ around C.
   (c) Find the flux of $\nabla \times \mathbf{F}$. (Verify Stokes' theorem directly in this case.)

4. (Problem 7, p. 547) Compute $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ where $\mathbf{F} = (x, y, z) \times (1, 1, 1)$ and S is the portion of the surface of the unit sphere outside the plane $x + y + z \geq 1$, with an outward-pointing normal. (Hint: the flux of the curl is the same over any other capping surface with the same intersection with the plane—which includes the portion of the plane itself within the intersecting circle.)