In-class exercises, April 14, 2009

1. Let S be the sphere of radius ρ centered at the origin. Prove that $(x,y,z)/\rho$ is a normal vector to S, as follows.

   (a) Consider the usual spherical coordinates:

   $$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

   This defines the parametrization $\Phi(\theta,\phi)$. Compute $\mathbf{T}_\theta = \partial\Phi/\partial\theta$, $\mathbf{T}_\phi = \partial\Phi/\partial\phi$, and $\mathbf{T}_\theta \times \mathbf{T}_\phi$. Finally, compute $\|\mathbf{T}_\theta \times \mathbf{T}_\phi\|$.

   (b) Show that $\mathbf{T}_\theta \times \mathbf{T}_\phi = (\rho \sin\phi)(-x,-y,-z)$, which must be normal to S. (Given any nonzero vectors **a** and **b**, $\mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b**.)

   (c) In this parametrization, the *unit* normal to the surface of S is

   $$\mathbf{n} = \frac{\mathbf{T}_\theta \times \mathbf{T}_\phi}{\|\mathbf{T}_\theta \times \mathbf{T}_\phi\|}.$$

   Show that

   $$\mathbf{n} = -\frac{1}{\rho}(x,y,z)$$

   and conclude that Φ is an orientation-reversing parametrization (since the normal vector points inward). This implies that $(x,y,z)/\rho$ is an outward-pointing unit normal vector to S.

2. This is a followup to the in-class discussion problem yesterday.

   (a) Find the flux of the vector field $\mathbf{F} = (2x,2y,2z)$ through the sides of the cylinder given by $\Phi(\theta,z) = (\cos\theta, \sin\theta, z)$ for $0 \le \theta \le 2\pi$ and $0 \le z \le 1$. Do the problem by computing $\mathbf{T}_\theta$, $\mathbf{T}_z$, and $\mathbf{T}_\theta \times \mathbf{T}_z$ explicitly to form $\iint_C \mathbf{F} \cdot d\mathbf{S}$.

   (b) Show that an equivalent approach is to take $\mathbf{n} = (x,y,0)$, so that Cavalieri's principle can be used to compute $\iint_C \mathbf{F} \cdot \mathbf{n}\, dS$.

3. (Problem 5, p. 497) Evaluate $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the surface $x^2 + y^2 + 3z^2 = 1$, $z \le 0$, and $\mathbf{F}$ is the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + zx^3y^2\mathbf{k}$. (Let $\mathbf{n}$, the unit normal, be upward pointing.) Hint: S is the graph of a function, so you can parametrize it as $\Phi(x,y) = (x,y,g(x,y))$ for a suitable g. Then $\mathbf{T}_x = (1,0,\partial g/\partial x)$ and $\mathbf{T}_y = (0,1,\partial g/\partial y)$.