Limits in $\mathbf{R}^n$

The limit may be defined in $\mathbf{R}^n$ analogously to that on the real line:

> Suppose $f : \mathbf{R}^n \to \mathbf{R}^m$. We say that the limit of $\mathbf{f}(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ is $\mathbf{L}$ and write $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that
>
> $$\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon \quad \text{whenever} \quad \|\mathbf{x} - \mathbf{a}\| < \delta.$$

Here $\|\mathbf{x}\|$ denotes the usual norm (length) of $\mathbf{x}$:

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

The definition of limit implies that *the value of f must approach* $\mathbf{L}$ *regardless of the path along which* $\mathbf{x}$ *approaches* $\mathbf{a}$.

On the real line, there are only two approaches: from the left and from the right, and the limit L exists if and only if the left- and right-hand limits both exist and equal L. For a function $f : \mathbf{R}^2 \to \mathbf{R}$, the limit as (x,y) approaches, say, the origin exists only if we get the same limiting value regardless of the path we take to the origin.

Example 1. Start Maple and select the Maple input icon at the top, denoted `[>`. Plot the graph of the function

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$$

near the origin. (See Example 13 on p. 122 of the textbook.) Proceed as follows:

```
with(plots);    this only needs to be done once
f := sin(x^2+y^2)/(x^2+y^2);
plot3d(f, x= -1..1, y= -1..1);
```

This plots a surface that you can rotate using the mouse. Click "Plot" menu at the top and select "Axes" to add the axes of your choice.

This plot suggests that $f(x,y)$ is well-behaved near $(0,0)$ and that $\lim_{(x,y)\to(0,0)} f(x,y) = 1$. See Example 13 on p. 122 for an analytical discussion.

Example 2. Plot $g(x,y) = x^2/(x^2+y^2)$ in the same way. (See Example 15 on pp. 123–124 of the text.) This surface has some odd "kinks" in it, and the graph suggests that the limit of $g(x,y)$ as $(x,y) \to (0,0)$ does not exist. This can be seen analytically as well:

$$\begin{aligned} \lim_{(0,y)\to(0,0)} g(x,y) &= 0 \quad \text{holding } x = 0 \\ \lim_{(x,0)\to(0,0)} g(x,y) &= 1 \quad \text{holding } y = 0. \end{aligned}$$

Since we get different limiting values as we approach the origin along the coordinate axes, the limit at $(0,0)$ does not exist.

Exercises

Compute each of the following limits. If the limit does not exist, then so state. Use Maple as needed to visualize each surface.

- Problem 8, p. 125.

  (a) $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2-(x-y)^2}{xy}$

  (b) $\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y}$

  (c) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$

- Problem 10, p. 125.

  (a) $\lim_{(x,y)\to(0,0)} \frac{e^{xy}}{x+1}$

  (b) $\lim_{(x,y)\to(0,0)} \frac{\cos x - 1 - (x^2/2)}{x^4+y^4}$

  (c) $\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$