Answers to in-class exercises, April 28, 2009

1. To save space, we'll write $f_x = \partial f/\partial x$ and so forth. We have

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = (f_{yz} - f_{zy})\mathbf{i} - (f_{xz} - f_{zx})\mathbf{j} + (f_{xy} - f_{yx})\mathbf{k} = \mathbf{0}.$$

2. In a direct calculation, we may take $\mathbf{n} = (x, y, z)$ as the outward unit normal to the unit sphere S. Therefore,

$$\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iint_{\partial S} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{\partial S} (x^2 + y^2 + z^2)\, dS = 4\pi$$

since $x^2 + y^2 + z^2 = 1$ on ∂S and the resulting integral is just the surface area of the unit sphere.

To apply Gauss's theorem, first observe that $\nabla \cdot \mathbf{F} = 3$, so

$$\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iiint_S 3\, dV = 4\pi,$$

since the latter integral is just 3 times the volume of S.

3. (Problem 23 p. 549)

   (a) A normal to S is $\mathbf{k}$, and in the disk, $\mathbf{F} \cdot \mathbf{k} = z = 0$. Hence the flux is 0.

   (b) Parametrize C as $\mathbf{r}(t) = (\cos t, \sin t, 0)$, which gives

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (\cos^2 t, 2\cos t \sin t + \cos t, 0) \cdot (-\sin t, \cos t, 0)\, dt \\ &= \int_0^{2\pi} \cos^2 t(\sin t + 1)\, dt \\ &= \pi. \end{aligned}$$

   (c) We have $\nabla \times \mathbf{F} = (2y + 1)\mathbf{k}$, so polar coordinates gives

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S (2y + 1)\, dA = \int_0^{2\pi} \int_0^1 (2\sin\theta + 1) r\, dr\, d\theta = \pi.$$

4. (Problems 7–8, p. 547) We have $\mathbf{F} = (y-z, z-x, x-y)$ and $\nabla \times \mathbf{F} = (-2,-2,-2)$. The flux of $\nabla \times \mathbf{F}$ across the indicated portion of the sphere is the same as that across the circle in the plane $x+y+z=1$. A normal to the plane is $(1,1,1)$, so a *unit* normal is $\mathbf{n} = (1,1,1)/\sqrt{3}$. This gives

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S (-2,-2,-2) \cdot \mathbf{n}\, dS = \iint_S -\frac{6}{\sqrt{3}}\, dS.$$

The circle circumscribes the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Its center must be at $(1/3,1/3,1/3)$ and has radius $\sqrt{2/3}$, so its area is $2\pi/3$. Therefore,

$$\iint_S -\frac{6}{\sqrt{3}}\, dS = \frac{2\pi}{3} \times \frac{-6}{\sqrt{3}} = \frac{-4\pi}{\sqrt{3}}.$$