Answers to in-class exercises, April 21, 2009

1. (Problem 26, p. 530) A unit normal to S is $(x, y, 0)$ since the radius of S is 1. Therefore,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (x^2 + y^2)\, dS = \iint_S 1\, dS = 2\pi,$$

that is, the surface area of S.

2. (Problem 15, p. 530) Green's theorem yields the formula

$$A = \frac{1}{2} \oint_{\partial D} (-y, x) \cdot d\mathbf{r}$$

where ∂D is the boundary of the ellipse. Parametrize the ellipse by $\mathbf{r}(t) = (a\cos t, b\sin t)$. Then

$$A = \frac{1}{2} \int_0^{2\pi} (-b\sin t, a\cos t) \cdot (-a\sin t, b\cos t)\, dt = \pi ab.$$

3. (Problem 5, p. 497) By Stokes' theorem,

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

where the boundary is positively oriented (i.e., traversed counterclockwise). In this case, ∂S is the unit circle in the xy plane ($z = 0$), so the usual parametrization gives

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (\sin t, -\cos t, 0) \cdot (-\sin t, \cos t, 0)\, dt = -2\pi.$$