Answers to in-class exercises, April 14, 2009

1. A calculation gives

$$\begin{aligned}
\mathbf{T}_\theta &= (\rho\sin\phi, \rho\cos\phi\sin\theta, \rho\sin\phi) \\
\mathbf{T}_\phi &= (\rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \rho\sin\phi) \\
\mathbf{T}_\theta \times \mathbf{T}_\phi &= -\rho^2\left(\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\right) \\
\mathbf{T}_\theta \times \mathbf{T}_\phi &= \rho^2\sin\phi.
\end{aligned}$$

   Note that, since $0 \le \phi \le \pi$ in the usual parametrization, $\sin\phi \ge 0$ and so we may drop the absolute values in the last step.

   (b) A comparison with the usual spherical coordinates gives the result.

   (c) Clear from (b).

2. (a) We have $\mathbf{F} = (2\cos\theta, 2\sin\theta, z)$ and $\mathbf{T}_\theta \times \mathbf{T}_z = (2\cos\theta, 2\sin\theta, 0)$, so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^1 \int_0^{2\pi} 4\, d\theta\, dz = 8\pi.$$

   (b) The unit normal is $\mathbf{n} = \frac{1}{2}(x, y, 0)$, so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S (x^2+y^2)\, dS = \iint_S 2\, dS,$$

   that is, twice the surface area of the cylinder. Cavalieri's principle gives a surface area of $2\pi \times 2 = 4\pi$, so $\iint_S \mathbf{F}\cdot d\mathbf{S} = 8\pi$ as in the previous calculation.

3. (Problem 5, p. 497) We have $z = -\sqrt{(1-x^2-y^2)/3}$ and $\nabla\times\mathbf{F} = (2x^3yz, -3x^2y^2z, -2)$. This parametrization gives

$$\mathbf{T}_x \times \mathbf{T}_y = \left(-\frac{x}{z\sqrt{3}}, -\frac{y}{z,\sqrt{3}}, 1\right)$$

   so that

$$\begin{aligned}
\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} &= \iint_S \left(-x^4y\sqrt{3} + x^2y^3\sqrt{3} - 2\right) dA \\
&= \int_0^{2\pi}\int_0^1 r\left(-\sqrt{3}\,r^5\cos^4\theta\sin\theta + \sqrt{3}\,r^5\cos^2\theta\sin^3\theta - 2\right) dr\, d\theta \\
&= -2\pi.
\end{aligned}$$