Answers to in-class exercises, April 7, 2009

1. (Problem 10, p. 418) Use Cavalieri's principle. The cone and the paraboloid meet where $6-r^2=r$, i.e., $r=2$. The volume is the sum of the volume of the cone from $z=0$ to $z=2$ and the volume of the paraboloic from $z=6$ down to $z=2$. This gives

$$\begin{aligned} V_{\text{cone}} &= \int_0^2 \pi r^2\,dr = \tfrac{8}{3}\pi \\ V_{\text{paraboloid}} &= \int_2^6 \pi(6-z)\,dz = 8\pi. \end{aligned}$$

   Thus the volume of the entire region is $\frac{32}{3}\pi$.

2. (Problem 21, p. 419) By symmetry, $\bar{x}=\bar{y}=0$. If the mass density is 1, then

$$\bar{z} = \frac{1}{V}\iiint_H z\,dV$$

   where V is the volume (mass) of the hemisphere H. Using spherical coordinates, we have

$$V = \iiint_H dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2a^3\pi}{3}$$

   and

$$\iiint_H z\,dV \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = \frac{a^4\pi}{4},$$

   so $\bar{z}=3a/8$.

3. (Problem 4, p. 447) (a) If, at every point $\mathbf{r}(t)$, the vector field $\mathbf{F}$ is perpendicular to the path, then $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=0$, which is the integrand of $\int_C \mathbf{F}\cdot d\mathbf{r}$.

   (b) Suppose C is parametrized by $\mathbf{r}(t)$, $a\le t\le b$. If $\mathbf{F}$ is parallel to $\mathbf{r}(t)$ at each point of C, then

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt \\
&= \int_a^b \mathbf{F}\cdot\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\,\|\mathbf{r}'(t)\|\,dt \\
&= \int_C \mathbf{F}\cdot\mathbf{T}\,ds \\
&= \int_C \|\mathbf{F}\|\,\|\mathbf{T}\|\cos\theta\,ds \\
&= \int_C \|\mathbf{F}\|\,ds
\end{aligned}$$

since $\mathbf{T}$ is a unit vector and $\theta = 0$ at every point of C.

4. (Problem 28, p. 367)

$$\int_1^4\int_1^{\sqrt{x}}(x^2+y^2)\,dy\,dx = \int_1^2\int_{y^2}^4(x^2+y^2)\,dx\,dy = \frac{1934}{105}.$$