Answers to in-class exercises, March 31, 2009

1. (Problem 3b, p. 427) Given $\mathbf{r}(t) = (t, 3t, 2t)$, we have $ds = \|\mathbf{r}'(t)\|\, dt = \sqrt{14}\, dt$. Thus
$$\int_C yz\, ds = \sqrt{14}\int_1^3 6t^2\, dt = 52\sqrt{14}.$$

2. (Problem 2d, p. 447) The parabola may be parametrized as $\mathbf{r}(t) = (t, 0, t^2)$ for $-1 \le t \le 1$. Therefore,
$$\int_C x^2\, dx - xy\, dy + dz = \int_{-1}^{1}(t^2, 0, 1)\cdot(1, 0, 2t)\, dt = \int_{-1}^{1}(t^2 + 2t)\, dt = \tfrac{2}{3}.$$

3. (Problem 12, p. 448) Proceed counterclockwise along each edge of the square S. The right-hand edge C_R can be parametrized as $(1, t)$ for $-1 \le t \le 1$. Then
$$\int_{C_R} \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}(2t, 1)\cdot(0, 1)\, dt = 2.$$
Similarly, the top edge C_T can be parametrized as $(t, 1)$ for $1 \ge t \ge -1$ and so
$$\int_{C_T} \mathbf{F}\cdot d\mathbf{r} = \int_{1}^{-1}(2t, t^2)\cdot(1, 0)\, dt = 0.$$
Along the left edge C_L, $\mathbf{r}(t) = (-1, t)$, $1 \ge t \ge -1$, so
$$\int_{C_L} \mathbf{F}\cdot d\mathbf{r} = \int_{1}^{-1}(-2t, 1)\cdot(0, 1)\, dt = -2,$$
and along the bottom edge C_B, $\mathbf{r}(t) = (t, -1)$, $-1 \le t \le 1$, so
$$\int_{C_B} \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}(-2t, t^2)\cdot(1, 0)\, dt = 0.$$
The integral $\oint_S \mathbf{F}\cdot d\mathbf{r}$ is the sum of these, which is 0.

4. (Problem 8, p. 405) We may assume, for purposes of choosing coordinates for parametrization, that the wire punctures the xy plane at the origin. Then the circle C may be parametrized as $\mathbf{r}(t) = (r\cos t, r\sin t)$ for $0 \le t \le 2\pi$. The unit tangent is
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = (-\sin t, \cos t).$$
Therefore, $\mathbf{H} = H\mathbf{T} = (-H\sin t, H\cos t)$, and so by Ampère Law,
$$I = \oint_C \mathbf{H}\cdot d\mathbf{r} = \int_0^{2\pi}(-H\sin t, H\cos t)\cdot(-r\sin t, r\cos t)\, dt = 2\pi H r,$$
which gives the result.