Answers to in-class exercises, March 17, 2009

1. (Problem 2, p. 404) The average value of f is
$$\frac{1}{A}\iint_D f\,dA,$$
where A is the area of the domain D of interest. In this case, D is a right triangle whose area is $A = 1/2$. The hypotenuse is the portion of the line $y = 1 - x$ in the first quadrant and it is equally easy to integrate in either order. Here
$$\begin{aligned}[f]_{\text{av}} &= 2\int_0^1\int_0^{1-x} e^x e^y\,dy\,dx \\ &= 2\int_0^1 (e - e^x)\,dx \\ &= 2.\end{aligned}$$

2. (Problem 9, p. 404) The total mass of the region, assuming the density to be 1, is
$$\begin{aligned}M &= \int_0^2\int_0^{2-x}\int_0^{2-x-y} 1\,dz\,dy\,dx \\ &= \int_0^2\int_0^{2-x}(2-x-y)\,dy\,dx \\ &= \int_0^2 (2-2x-\tfrac{1}{2}x^2)\,dx \\ &= \frac{4}{3}.\end{aligned}$$
Now compute each coordinate of the center of mass:
$$\begin{aligned}\bar{x} &= \frac{1}{M}\int_0^2\int_0^{2-x}\int_0^{2-x-y} x\,dz\,dy\,dx \\ &= \frac{3}{4}\int_0^2\int_0^{2-x} x(2-x-y)\,dy\,dx \\ &= \frac{3}{4}\int_0^2\int_0^{2-x}\frac{x(2-x)^2}{2}\,dx \\ &= \frac{1}{2}.\end{aligned}$$

By symmetry, $\bar{y} = \bar{z} = \bar{x}$ and so the center of mass is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

3. (Problem 10, p. 405) The idea is exactly the same as in the previous problem, but use cylindrical coordinates. The mass of the cylinder is

$$M = \iiint (x^2 + y^2) z^2 \, dV = \int_1^2 \int_0^{2\pi} \int_0^1 (r^2 z^2) r \, dr \, d\theta \, dz = \frac{7\pi}{6}.$$

The circular symmetry of the cylinder and the density implies that $\bar{x} = \bar{y} = 0$. We have

$$\bar{z} = \frac{1}{M} \int_1^2 \int_0^{2\pi} \int_0^1 z(r^2 z^2) r \, dr \, d\theta \, dz = \left(\frac{6}{7\pi}\right)\left(\frac{15\pi}{8}\right) = \frac{45}{28}.$$

4. (Problem 15, p. 405) The discussion in the text implies that for the purposes of these types of calculations, all the mass can be treated as though it were concentrated at a point (i.e., at the center of mass of each object). Therefore, if we have an object of mass M at location $\mathbf{X}$ and another of mass m at location $\mathbf{x}$, then the gravitational potential between them is

$$\frac{GmM}{\|\mathbf{x} - \mathbf{X}\|},$$

where G is the gravitational constant. Since $G = 6.67 \times 10^{-11}$ Nm^2/kg^2 and $M = 3 \times 10^{26}$ kg, the gravitational potential to three significant digits is

$$\frac{(6.67 \times 10^{-11}) \times (3 \times 10^{26}) m}{2 \times 10^8} \text{J/kg} = (1.00 \times 10^8) m \text{ J/kg}.$$