Answers to in-class exercises, Feb. 24, 2009

These problems are from Sections 5.4 and 5.5.

1. (Problem 10, p. 354) If we integrate with respect to y first, then we need to break up the integral into two parts: one from $0 \le x \le 1$ and the other from $1 \le x \le 2$ to accommodate the two different sides of the triangle. The first integral is

$$\int_0^1 \int_x^{3x} e^{x-y}\, dy\, dx = \int_0^1 \left(1 - e^{-2x}\right) dx = \tfrac{1}{2}(1 + e^{-2}).$$

   The second integral is

$$\int_1^2 \int_x^{4-x} e^{x-y}\, dy\, dx = \int_1^2 \left(1 - e^{2x-4}\right) dx = \tfrac{1}{2}(1 + e^{-2}).$$

   The integral over the whole region therefore is $1 + e^{-2}$.

2. (Problem 12, p. 354) The region is the wedge between the parabola $y = x^2$ and the line $y = x$ between $x = 0$ and $x = 1$. This implies

$$\int_0^1 \int_{x^2}^{x} f(x,y)\, dy\, dx = \int_0^1 \int_y^{\sqrt{y}} f(x,y)\, dx\, dy.$$

3. (Problem 9, p. 364) The outer boundary of the region is defined by the ellipse $2x^2 + 3y^2 = 10$. By symmetry, we may compute the volume in the first octant and multiply the result by 4. This gives

$$\begin{aligned}
\tfrac{1}{4}V &= \int_0^{\sqrt{5}} \int_0^{[(10-2x^2)/3]^{1/2}} \int_{x^2+y^2}^{10-x^2-2y^2} 1\, dz\, dy\, dx \\
&= \int_0^{\sqrt{5}} \int_0^{[(10-2x^2)/3]^{1/2}} (10 - 2x^2 - 3y^2)\, dy\, dx \\
&= \int_0^{\sqrt{5}} \left[\tfrac{1}{3}(10-2x^2)^{3/2} - \tfrac{1}{27}(10-2x^2)^{1/2}\right] dx \\
&= \frac{25}{12}\pi\sqrt{6}.
\end{aligned}$$

   Hence

$$V = \frac{100\pi\sqrt{6}}{12} = \frac{50\pi}{\sqrt{6}}.$$

4. (Problem 15, p. 364) We have

$$\begin{aligned}
\iiint_D (x^2 + y^2 + z^2)\, dV &= \int_0^a \int_0^{a-y} \int_0^{a-x-y} (x^2 + y^2 + z^2)\, dz\, dx\, dy \\
&= \frac{a^5}{20}.
\end{aligned}$$