# Answers to in-class exercises, Feb. 10, 2009

1. (Problem 1, p. 222) $f(x,y) = x^2 - y^2 + xy$. The critical points occur where $\nabla f = (2x+y, x-2y) = (0,0)$. The only possibility is the origin, where $f = 0$. However, inspection reveals points near the origin where $f > 0$ and $f < 0$, so the origin must be a saddle point. Alternatively, notice that $D = f_{xx}f_{yy} - f_{xy}^2 = -5$ everywhere; this analytical criterion implies that the origin is a saddle point.

2. (Problem 7, p. 222) $f(x,y) = 3x^2 + 2xy + 2x + y^2 + y + 4$. We have $\nabla f = (6x + 2y + 2, 2x + 2y) = (0,0)$ at $(-1/4, -1/4)$. Now $D = f_{xx}f_{yy} - f_{xy}^2 = 8$ everywhere, and since $f_{xx} = 6 > 0$ everywhere, the critical point is a minimum.

3. (Problem 17, p. 223) $f(x,y) = (x^3 + 3y^2)e^{1-x^2-y^2}$. We have

$$\nabla f = e^{1-x^2-y^2} \cdot \left(3x^2 - 2x^4 + 3xy^2, 6y - 2(x^3y + 3y^3)\right),$$

   which is zero at the following points:

$$(0,0), \quad (0,\pm 1), \quad \text{and} \quad \left(\pm\tfrac{1}{2}\sqrt{6}, 0\right).$$

   The discriminant of $f$ is 0 at $(0,0)$, which means that the critical point cannot be classified by the usual methods—but notice that, since $e^{1-x^2-y^2} > 0$, there are points near the origin at which $f$ assumes both negative and positive values (so the origin must be a saddle). The discriminant of $f$ is 72 at $(0,\pm 1)$, and since $f_{xx}(0,\pm 1) = -6 < 0$, this point represents a local maximum. The discriminant of $f$ is 72 at $\left(\pm\frac{1}{2}\sqrt{6}, 0\right)$, and since $f_{xx}\left(\pm\frac{1}{2}\sqrt{6}, 0\right) = 3\sqrt{6}e^{-1/2} > 0$, this point represents a local minimum.

4. (Problem 19, p. 223) Let $f(x,y) = (y - 3x^2)(y - x^2)$.

   (a) We have $\nabla f = \left(-6x(y-x^2) - 2(y-3x^2)x, 2y - 4x^2\right)$, so $(0,0)$ is a critical point by inspection.

   (b) Let $h(t) = f(\mathbf{c}(t))$, where $\mathbf{c}(t) = (at, bt)$. Apply the chain rule to $h'(t)$ or substitute $x = at$ and $y = bt$ into the formula for $f$ and differentiate directly to show that $x = y = t = 0$ is a critical point. Take another derivative and simplify to show that $h''(t) = 2b^2$, which is always positive if $b \neq 0$. Hence $t = 0$ is always a relative minimum in the graph of $f$ when we travel on a straight line through the origin.

   (c) Consider a path through the origin along a parabola, e.g., $y = ax^2$. Then $f(x, ax^2) = x^4(a-3)(a-1)$, which is negative for $1 < a < 3$. Hence there are points arbitrarily close to the origin at which $f < 0$.