Answers to Exam 3

1. By symmetry, it suffices to compute the integral over the top half of the unit circle. Polar coordinates give

$$\int_0^{\pi} \int_0^1 re^{-r^2}\, dr\, d\theta = \frac{\pi}{2}\left(1 - e^{-1}\right).$$

2. **Method 1.** The volume consists of the region between the paraboloid $z = r^2$ below and the sphere of radius $\sqrt{2}$ above. The two surfaces intersect when $r^2 = \sqrt{2 - r^2}$, i.e., $r = 1$. Thus the enclosed volume is

$$\iiint_V dV = \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r\, dz\, dr\, d\theta = \frac{4\pi\sqrt{2}}{3} - \frac{7\pi}{6}.$$

Method 2. Use Cavalieri's principle. The area of a cross-section at level z of the lower paraboloid is $A(z) = \pi z$ so

$$V_{\text{lower}} = \int_0^1 \pi z\, dz = \frac{\pi}{2}.$$

Since $r^2 + z^2 = 2$, the area of a cross-section of the top part is $A(z) = \pi(2 - z^2)$, so that

$$V_{\text{upper}} = \int_1^{\sqrt{2}} \pi(2 - z^2)\, dz = \frac{\pi}{3}\left(4\sqrt{2} - 5\right).$$

The total volume is $V_{\text{lower}} + V_{\text{upper}} = \frac{4}{3}\pi\sqrt{2} - \frac{7}{6}\pi$ as before.

3. The integral is defined as

$$\lim_{b \searrow 0} \lim_{a \searrow 0} \int_b^1 \int_a^1 \frac{dx\, dy}{\sqrt{xy}} = \lim_{b \searrow 0} \lim_{a \searrow 0} \left(2 - 2b^{1/2}\right)\left(2 - 2a^{1/2}\right) = 4.$$

4. (a) Parametrize the arc as $\mathbf{r}(t) = (t, t^2)$ for $-1 \le t \le 2$. The work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{2} (t, t^2) \cdot (1, 2t)\, dt = \int_{-1}^{2} (t + 2t^3)\, dt = 9.$$

(b) We have

$$\begin{aligned}
\int_C \mathbf{T} \cdot d\mathbf{r} &= \int_a^b \mathbf{T}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \cdot \mathbf{r}'(t)\, dt \quad \text{by definition of } \mathbf{T} \\
&= \int_a^b \|\mathbf{r}'(t)\|\; dt \quad \text{since } \mathbf{r}' \cdot \mathbf{r}' = \|\mathbf{r}'\|^2 \\
&= \int_C ds \\
&= \text{length of } C.
\end{aligned}$$