Answers to Exam 2

1. We have
$$\nabla f = (6x^2 + y^2 + 10x, 2xy + 2y).$$
One obvious critical point is $(0,0)$. Using the second component, we check whether there are any other critical points along $y = 0$; the first component gives $6x^2 + 10x = x(6x + 10) = 0$ when $x = -5/3$. Hence $(-5/3, 0)$ is another critical point. The second component of ∇f is also 0 when $x = -1$, so substituting into the first component implies $6 - 10 + y^2 = 0$, i.e., $y = \pm 2$. Thus the other critical points are $(-1, \pm 2)$.

   To classify each point, we calculate the Hessian ($f_{xx}f_{yy} - f_{xy}^2$) and the second derivative f_{xx}. This yields the following table:

point	f	Hessian	f_{xx}	classification
$(0,0)$	0	20	10	relative minimum
$(-5/3,0)$	$125/27 \approx 4.63$	$40/3$	-10	relative maximum
$(-1,\pm 2)$	3	-16	—	saddle

   The first two, of course, are the minimum and maximum values of f.

2. There was a typo on the exam: it should have read $z^2 = a^2(x^2 + y^2)$ since then $z = a\sqrt{x^2 + y^2} = ar$, which gives a 45° cone. As written, the object $z = a^2(x^2 + y^2)$ is a paraboloid, not a cone, but in either case we can apply Cavalieri's principle in the same way.

   Consider the paraboloid. We slice horizontally and consider the volume $dV = A(z)\,dz$ generated by the corresponding disk at level z. Since $z = a^2r^2$, we have $A(z) = \pi(z/a^2)$, so the volume of the section is
$$V = \int_1^2 A(z)\,dz = \int_1^2 \frac{\pi z}{a^2}\,dz = \frac{3\pi}{2a^2}.$$

3. We must minimize $f(x, y) = x^2 + y^2$ subject to the constraint $g = x + y = 20$. We look for points where
$$\nabla f = (2x, 2y) = \lambda \nabla g = \lambda(1, 1).$$
The minimum therefore occurs where $x = y$, i.e., $x = y = 10$, where $f = 200$.

4. Sketch the region of integration. In the xy plane, the region of integration extends above the parabola $y = x^2$ and stops at $y = 1$. The plane $y + z = 1$ bounds the top of the region, and the xy plane is the bottom. When we integrate with respect to x first, we sweep across the two arms of the parabola: $-\sqrt{y} \leq x \leq \sqrt{y}$. Also, y goes from 0 to the plane, so $0 \leq y \leq 1 - z$, and $0 \leq z \leq 1$. Thus

$$\int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f \, dz \, dy \, dx = \int_{0}^{1} \int_{0}^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f \, dx \, dy \, dz.$$