Answers to Exam 1

1. I gave full credit for answers that included the portion on the xy plane as well as those that included only points strictly above the xy plane.

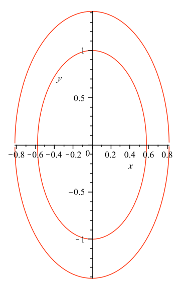
   (a) Two inequalities are needed: $x^2 + y^2 + z^2 \leq 4$ and $z > 0$ (or $z \geq 0$, which includes the points on the xy plane).

   (b) $\rho \leq 2$ and $0 \leq \phi < \pi/2$ or ($0 \leq \phi \leq \pi/2$ if you include the equator, i.e., points on the xy plane).

   (c) $r^2 + z^2 \leq 4$ and $z > 0$ (or $z \geq 0$).

2. Given $f(x,y) = 3x^2 + y^2$.

   (a) Level curves of $f = 1$ and $f = 2$ are illustrated below.



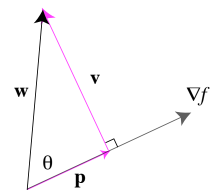
   (b) Use the linearization about the point $(x_0, y_0) = (1, -1)$:

$$\begin{aligned} f(x,y) &\approx f(x_0,y_0) + \nabla f(x_0,y_0)\cdot(x-x_0, y-y_0) \\ &= 4 + (6x, 2y)\big|_{(1,-1)} \cdot (x-1, y+1) \\ &= 4 + (6,-2)\cdot(-0.03, -0.02) \\ &= 3.86. \end{aligned}$$

   The "true" value is $f(0.97, -1.02) = 3.8631$.

   (c) One direction is $\nabla f(1,1) = (6,2)$ (or any positive multiple thereof).

   (d) Here's one way to solve the problem. We have $\nabla f(1,1) = (6,2)$. The direction $\mathbf{v} = (-2,6)$ perpendicular to ∇f is the direction in which f is constant. So we need to find the direction $\mathbf{w}$ whose projection $\mathbf{p} = \frac{1}{2}\nabla f(1,1) = (3,1)$. This can be accomplished if $\mathbf{w}$, $\mathbf{v}$, and $\mathbf{p}$ form



a 60°-30°-90° triangle as illustrated. If we let $\|\mathbf{w}\| = \sqrt{40}$, then since $\|\mathbf{p}\| = \sqrt{10}$, we need $\|\mathbf{v}\| = \sqrt{30}$. Let $\mathbf{u}$ be the unit vector in the perpendicular direction to $\nabla f(1,1) = (6,2)$. Then

$$\mathbf{v} = \sqrt{30}\,\mathbf{u} = \sqrt{30}\,\frac{(-2,6)}{\sqrt{40}} = \frac{(-1,3)}{\sqrt{3}}.$$

Thus

$$\mathbf{w} = \mathbf{p} + \mathbf{v} = (3,1) + \frac{(-1,3)}{\sqrt{3}}$$

(or any nonzero multiple thereof) is a direction in which f increases at half its maximum rate.

(e) Use the chain rule:

$$\left.\frac{d}{dt} f(\mathbf{c}(t))\right|_{t=1} = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\big|_{t=1} = (6,2) \cdot \left(1, \tfrac{1}{2}\right) = 7.$$

3. Given $\mathbf{a} = (2,1,-3)$ and $\mathbf{b} = (-1,1,1)$.

(a) Use the dot product formula to find θ:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \frac{-4}{\sqrt{42}}$$

so $\theta = \cos^{-1}\left(-4/\sqrt{42}\right)$.

(b) Let $\mathbf{u} = \mathbf{a}/\sqrt{14}$ be the unit vector in the same direction as $\mathbf{a}$. Then the required projection is

$$|\mathbf{b} \cdot \mathbf{u}|\,\mathbf{u} = \tfrac{2}{7}\mathbf{a}.$$

(c) We have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -3 \\ -1 & 1 & 1 \end{vmatrix} = (4,1,3).$$

4. Show that the cosine of the angle between $\mathbf{a}$ and $\mathbf{v}$ equals the cosine of the angle between $\mathbf{v}$ and $\mathbf{b}$. Since $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$, we have

$$\cos \theta_{\mathbf{av}} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{a}\| \, \|\mathbf{v}\|} = \frac{\|\mathbf{a}\|(\mathbf{a} \cdot \mathbf{b}) + \|\mathbf{b}\| \, \|\mathbf{a}\|^2}{\|\mathbf{a}\| \, \|\mathbf{v}\|} = \frac{(\mathbf{a} \cdot \mathbf{b}) + \|\mathbf{b}\| \, \|\mathbf{a}\|}{\|\mathbf{v}\|}$$

and

$$\cos \theta_{\mathbf{vb}} = \frac{\mathbf{b} \cdot \mathbf{v}}{\|\mathbf{b}\| \, \|\mathbf{v}\|} = \frac{\|\mathbf{a}\| \, \|\mathbf{b}\|^2 + \|\mathbf{b}\|(\mathbf{a} \cdot \mathbf{b})}{\|\mathbf{b}\| \, \|\mathbf{v}\|} = \frac{\|\mathbf{b}\| \, \|\mathbf{a}\| + (\mathbf{a} \cdot \mathbf{b})}{\|\mathbf{v}\|},$$

which proves the result.