Homework answers, week of April 13, 2009

Section 7.6 (p. 497)

6. A calculation shows $\nabla \times \mathbf{F} = (0, -2z, 3y - 1)$. Since the hemisphere of interest has radius 4, the usual spherical coordinates

$$x = 4\sin\phi\cos\theta, \quad y = 4\sin\phi\sin\theta, \quad z = 4\cos\phi$$

give $\mathbf{F} = (0, -8\cos\phi, 12\sin\phi\sin\theta - 1)$ and the surface normal

$$\mathbf{T}_\theta \times \mathbf{T}_\phi = -16\sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\theta),$$

which points inward. Thus, we take the negative of this expression for the outward-pointing normal. Therefore,

$$\begin{aligned} \iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2} \mathbf{F}\cdot(-\mathbf{T}_\theta \times \mathbf{T}_\phi)\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2} 16\sin\phi\cos\phi(4\sin\phi\sin\theta - 1)\,d\phi\,d\theta \\ &= -16\pi. \end{aligned}$$

Note: Stokes' theorem implies that this integral must equal $\int_{\partial S}\mathbf{F}\cdot d\mathbf{r}$, where ∂S is the boundary of the hemisphere, i.e., the circle of radius 4 in the xy plane. The theorem can be used to double-check the original computation.

10. A unit vector to each point of this surface is $(x, y, 0)$, so that $\mathbf{F}\cdot\mathbf{n} = x + y$, which gives

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^1\int_0^{2\pi}(\cos\theta + \sin\theta)\,d\theta\,dz = 0.$$

16. (a) A calculation (or see formula (4) on p. 495) shows that

$$\mathbf{T}_x \times \mathbf{T}_y = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right)$$

so that $\mathbf{F}\cdot(\mathbf{T}_x \times \mathbf{T}_y) = -1$. Therefore, since S the graph of a function over the unit disk,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot(\mathbf{T}_x \times \mathbf{T}_y)\,dx\,dy = \int_0^{2\pi}\int_0^1 -r\,dr\,d\theta = -\pi.$$

(b) We have

$$\mathbf{F}\cdot(\mathbf{T}_x\times\mathbf{T}_y)=\frac{x}{\sqrt{2(x^2+y^2)}}-\frac{1}{\sqrt{2}},$$

so that

$$\iint_S \mathbf{F}\cdot d\mathbf{S}=\int_0^1\int_0^{2\pi}\left(\frac{\cos\theta}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)r\,d\theta\,dr=-\frac{\pi}{\sqrt{2}}.$$

Section 8.1 (p. 528)

2. Parametrize the disk as $\mathbf{r}(\theta)=(D\cos\theta,D\sin\theta)$. Theorem 2 states that the area enclosed by ∂D is $\frac{1}{2}\int_{\partial D}\mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}=-y\mathbf{i}+x\mathbf{j}$. Hence,

$$A=\frac{1}{2}\int_{\partial D}(-y,x)\cdot d\mathbf{r}=\frac{1}{2}\int_0^{2\pi}(-D\sin\theta,D\cos\theta)\cdot(-D\sin\theta,D\cos\theta)\,d\theta=\pi D^2.$$

10. The vector field is $\mathbf{F}=P\mathbf{i}+Q\mathbf{j}=(\partial f/\partial y)\mathbf{i}-(\partial f/\partial x)\mathbf{j}$ on D. By Green's theorem,

$$\int_{\partial D}\mathbf{F}\cdot d\mathbf{r}=\iint_D\left(-\frac{\partial}{\partial x}\frac{\partial f}{\partial x}-\frac{\partial}{\partial y}\frac{\partial f}{\partial y}\right)dA=0$$

since f is harmonic.