Homework answers, week of March 16, 2009

Section 6.4 (pp. 415–416)

4. We have

$$\begin{aligned}\int_0^1 \int_0^{e^y} \log x\,dx\,dy &= \lim_{a\searrow 0} \int_0^1 \int_a^{e^y} \log x\,dx\,dy \\ &= \lim_{a\searrow 0} \int_0^1 [a(1+\log a) + e^y(y-1)]\,dy \\ &= \lim_{a\searrow 0} a(1+\log a) + (2-e) \\ &= 2-e \quad \text{by l'Hôpital's rule.}\end{aligned}$$

5. (a) We have

$$\begin{aligned}\iint_D \frac{dA}{(x^2+y^2)^{2/3}} &= \int_0^{2\pi} \int_0^1 \frac{r\,dr\,d\theta}{r^{4/3}} \\ &= \lim_{a\searrow 0} \int_0^{2\pi} \int_a^1 r^{-1/3}\,dr\,d\theta \\ &= \lim_{a\searrow 0} 3\pi\left(1-a^{2/3}\right) \\ &= 3\pi.\end{aligned}$$

(b) Formally, if $\lambda \neq 1$, we have

$$\iint_D \frac{dA}{(x^2+y^2)^{\lambda}} = \lim_{a\searrow 0} \int_0^{2\pi} \int_a^1 \frac{r\,dr\,d\theta}{r^{2\lambda}} = 2\pi \lim_{a\searrow 0} \frac{a^{2(1-\lambda)}-1}{2(\lambda-1)}.$$

The limit exists for $\lambda < 1$ and does not exist for $\lambda > 1$. If $\lambda = 1$, then the innermost integral becomes

$$\lim_{a\searrow 0} \int_a^1 \frac{dr}{r} = \lim_{a\searrow 0} (-\log a),$$

which does not exist. Hence the integral exists for $\lambda < 1$.

6. For the y-simple region in the figure, we may define

$$\iint_D f\,dA = \lim_{b\to\infty}\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f\,dy\,dx$$

whenever the limit exists. This gives

$$\begin{aligned}
\iint_D xye^{-(x^2+y^2)}\,dA &= \lim_{b\to\infty}\int_0^b \int_0^1 xye^{-(x^2+y^2)}\,dy\,dx \\
&= \lim_{b\to\infty}\int_0^b \frac{xe^{-x^2} - xe^{1-x^2}}{2}\,dx \\
&= \lim_{b\to\infty}\frac{1-e^{-1}-e^{-b^2}+e^{-b^2-1}}{4} \\
&= \frac{1-e^{-1}}{4}.
\end{aligned}$$

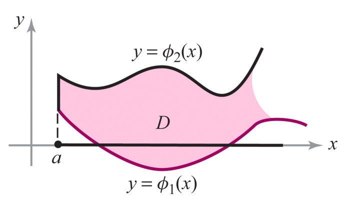

Figure 1: For Problem 6.

10. Divide the unit square D diagonally along the line $y = x$. In the upper left portion, we have

$$\begin{aligned}
\iint_D \frac{dA}{\sqrt{|x-y|}} &= \int_0^1 \int_x^1 \frac{dy\,dx}{\sqrt{y-x}} \\
&= \int_0^1 \int_0^{1-x} \frac{du\,dx}{\sqrt{u}} \\
&= \int_0^1 2(1-x)^{1/2}\,dx \\
&= 4/3.
\end{aligned}$$

Along the lower portion, a similar calculation gives

$$\begin{aligned}\iint_D \frac{dA}{\sqrt{|x-y|}} &= \int_0^1 \int_0^x \frac{dy\,dx}{\sqrt{x-y}} \\ &= \int_0^1 2x^{1/2}\,dx \\ &= 4/3.\end{aligned}$$

Thus,

$$\iint_D \frac{dA}{\sqrt{|x-y|}} = \frac{8}{3}.$$

Section 4.1 (p. 274)

15. From Kepler's Law, the period T of the satellite is determined from the relation

$$T = \left(\frac{4\pi^2 r^3}{GM}\right)^{1/2}.$$

Given that the mass of the earth is $M = 5.98 \times 10^{24}$ kg, $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$, and $r = 4500$ mi $= 7.24 \times 10^6$ m, we have $T \approx 6130$ s.

16. The centripetal acceleration is

$$\|\mathbf{r}''\| = \frac{GM}{r^2} \approx 7.6 \text{ m/s}^2$$

and the centripetal force is about $7.6m$ N, where m is the mass of the satellite in kg, directed toward the center of the earth.

Section 4.2 (p. 282)

13. (a) The length of the arc is simply

$$\ell = \int_a^b ds = \int_a^b \|\mathbf{c}''(t)\|\,dt = b - a.$$

(b) The unit tangent is $\mathbf{T}(s) = \mathbf{c}'(s)/\|\mathbf{c}'(s)\| = \mathbf{c}'(s)$, so $\mathbf{T}'(s) = \mathbf{c}''(s)$, from which the result is immediate.