

Homework answers, week of Feb. 23, 2009

Section 5.3 (p. 348)

4. An ellipse centered at the origin is described by the equation $(x/a)^2 + (y/b)^2 = 1$. Therefore, the area is

$$\begin{aligned} A &= \int_{-a}^a \int_{-(b/a)\sqrt{a^2-x^2}}^{(b/a)\sqrt{a^2-x^2}} 1 \, dy \, dx \\ &= \frac{2b}{a} \int_{-a}^a \sqrt{a^2-x^2} \, dx. \end{aligned}$$

Make the substitution $x = a \sin \theta$; then $dx = a \cos \theta$. By symmetry, we may find the area of the upper half of the ellipse and double the result, so $\sqrt{a^2-x^2} = a \cos \theta$. Therefore, the integral becomes

$$\begin{aligned} A &= 4b \int_0^{\pi/2} a \cos^2 \theta \, d\theta \\ &= 4ab \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} \, d\theta \\ &= \pi ab. \end{aligned}$$

9. The region is the portion of the annulus above the y axis contained between the unit circle and the circle of radius $\sqrt{2}$, which is y -simple. The double integral is evaluated in three pieces as follows:

$$\iint_D f \, dA = \int_{-\sqrt{2}}^{-1} \int_0^{\sqrt{2-x^2}} f \, dy \, dx + \int_{-1}^1 \int_{\sqrt{1-x^2}}^{\sqrt{2-x^2}} f \, dy \, dx + \int_1^{\sqrt{2-x^2}} \int_0^{\sqrt{2-x^2}} f \, dy \, dx.$$

The first of these becomes

$$\int_{-\sqrt{2}}^{-1} \left(\sqrt{2-x^2} + \frac{1}{2}x(2-x^2) \right) dx = \frac{1}{4}\pi - \frac{5}{8}.$$

The second becomes

$$\int_{-1}^1 \left(\sqrt{2-x^2} - \sqrt{1-x^2} + \frac{1}{2}x \right) dx = 1,$$

and the third is

$$\int_{\sqrt{2}}^1 \left(\sqrt{2-x^2} + \frac{1}{2}x(2-x^2) \right) dx = \frac{3}{8} - \frac{1}{4}\pi.$$

The sum of these is the result, namely $\pi/2$.

You may evaluate these integrals with Maple or use a table of integrals, which gives

$$\int \sqrt{a^2-x^2} dx = \frac{1}{2} \left[\sqrt{a^2-x^2} + a^2 \sin^{-1} \left(\frac{x}{a} \right) \right]. \quad (1)$$

12. The simplest procedure is to use Cavalieri's principle. Slice the cone horizontally; the radius of the disk is r at $z = 0$ and decreases linearly with z to 0 at $z = h$. Therefore, $r(z) = r - (r/h)z$, so the area of the disk is at level z is $A(z) = \pi r^2(z)$. Hence the volume of the cone is

$$V = \int_0^h A(z) dz = \int_0^h \pi \left(r - \frac{rz}{h} \right) dz = \frac{\pi r^2 h}{3}.$$

Another possibility is to treat the "cap" of the cone as a surface and consider the volume between the cap and the xy plane. The surface is $z = h - (h/r)\sqrt{x^2+y^2}$, and given the circular symmetry, we can integrate over the portion that's in the first quadrant and multiply by 4. Thus

$$V = 4 \int_0^r \int_0^{\sqrt{r-x^2}} \left(h - \frac{h\sqrt{x^2+y^2}}{r} \right) dy dx.$$

Of course, this calculation is much easier in cylindrical coordinates (covered in Chapter 6), but just to illustrate, we recall that $dydx = r dr d\theta$ so taking the radius of the cone as R ,

$$V = \int_0^{2\pi} \int_0^R \left(h - \frac{hr}{R} \right) r dr d\theta = \frac{\pi R^2 h}{3},$$

as expected.

Review problems (p. 365)

10. For each fixed (x, y) , we integrate from $z = 0$ to the plane, which is $z = (2 - x - y)/2$. Given the symmetry, we may integrate over the first octant and multiply the result by 4 to get the volume. This gives

$$V = 4 \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}/2} (2-x-y)/2 \, dy \, dx = \pi - \sqrt{2}.$$

Equation (1) is helpful here.

22. The region is a tetrahedron whose corners are the origin, $(1, 0, 0)$, $(1, 1, 0)$ and $(1, 1, 1)$. We have

$$\int_0^y \int_0^x \int_0^1 f(x, y, z) \, dz \, dy \, dx = \int_0^1 \int_z^1 \int_y^1 f(x, y, z) \, dx \, dy \, dz.$$