

Homework answers, week of Feb. 16, 2009

Section 3.4 (pp. 244–245)

18. Use Lagrange multipliers to minimize $P(x,y) = x + 2y/\cos \theta$ subject to $A = xy + y^2 \tan \theta = \text{constant}$. (Remember that the angle θ is fixed.) It's convenient in this problem to write the Lagrange formulation as $\lambda \nabla P = \nabla A$, which implies $\lambda = y$ and $2y = x \cos \theta + 2y \sin \theta$, so that

$$x = 2y \left(\frac{1}{\cos \theta} - \tan \theta \right).$$

Therefore,

$$A \cos \theta = 2y^2 \left(\frac{1}{\cos \theta} - \tan \theta \right) \cos \theta + y^2 \sin \theta = y^2 (2 - \sin \theta),$$

which gives the result.

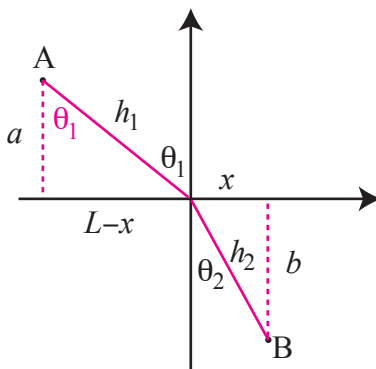


Figure 1: Snell's Law.

20. Figure 1 shows the setup. Light travels from A to the interface at the origin, then refracts and travels to B. Suppose that $B = (x, -b)$ and $A = (x - L, a)$

so that the horizontal separation between them is L . The respective right triangles imply

$$\tan \theta_1 = \frac{L-x}{a} \quad (1)$$

$$\tan \theta_2 = \frac{x}{b}. \quad (2)$$

The Pythagorean theorem and Eqs. (1)–(2) imply that

$$\begin{aligned} h_1 &= \sqrt{a^2 + (L-x)^2} = \sqrt{a^2 + a^2 \tan^2 \theta_1} = a \sec \theta_1 \\ h_2 &= \sqrt{b^2 + x^2} = \sqrt{b^2 + b^2 \tan^2 \theta_2} = b \sec \theta_2. \end{aligned}$$

Suppose the light ray travels along h_1 at speed v_1 and along h_2 at v_2 . The total time required is

$$T(\theta_1, \theta_2) = \frac{h_1}{v_1} + \frac{h_2}{v_2} = \frac{a \sec \theta_1}{v_1} + \frac{b \sec \theta_2}{v_2}. \quad (3)$$

Since the horizontal distance between A and B is fixed, Eqs. (1)–(2) imply that

$$g(\theta_1, \theta_2) = a \tan \theta_1 + b \tan \theta_2 = L. \quad (4)$$

Thus we need to minimize T subject to the constraint in Eq. (4). We solve $\nabla T = \lambda \nabla g$, i.e.,

$$\begin{aligned} \frac{a \sec \theta_1 \tan \theta_1}{v_1} &= \lambda a \sec^2 \theta_1 \\ \frac{b \sec \theta_2 \tan \theta_2}{v_2} &= \lambda b \sec^2 \theta_2 \end{aligned}$$

which implies

$$\frac{\tan \theta_1}{v_1 \sec \theta_1} = \lambda = \frac{\tan \theta_2}{v_2 \sec \theta_2},$$

which simplifies to

$$\frac{\sin \theta_1}{v_1} = \lambda = \frac{\sin \theta_2}{v_2}.$$

Hence the total travel time T is minimized when

$$\frac{v_1}{v_2} = \frac{\sin \theta_1}{\sin \theta_2},$$

which is Snell's Law.

23. As I mentioned in class, an analysis for the 2×2 case suffices, since the ideas are the same for \mathbb{R}^n . If \mathbf{A} is symmetric, then

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

and by writing out all the terms we find

$$f(\mathbf{x}) = \frac{1}{2}(\mathbf{A}\mathbf{x}) \cdot \mathbf{x} = \frac{1}{2}(ax_1^2 + 2bx_1x_2 + cx_2^2).$$

(Such a function is called a *quadratic form*.) Therefore,

$$\nabla f = \frac{1}{2} \begin{pmatrix} 2ax_1 + 2bx_2 \\ 2bx_1 + 2cx_2 \end{pmatrix} = \mathbf{A}\mathbf{x}.$$

To find the extrema of f subject to $g = x^2 + y^2 = 1$, we calculate

$$\nabla f = \mathbf{A}\mathbf{x} = \lambda \nabla g = 2\lambda \mathbf{x}.$$

The solution of this equation is an eigenvector of \mathbf{A} with eigenvalue $\mu = 2\lambda$, i.e., $\mathbf{A}\mathbf{x} = \mu\mathbf{x}$. If \mathbf{x} is an eigenvector of \mathbf{A} , then

$$f(\mathbf{x}) = \frac{1}{2}(\mu\mathbf{x}) \cdot \mathbf{x} = \frac{1}{2}\mu\|\mathbf{x}\|^2.$$

Depending on the sign of μ , this value is either a maximum or minimum of f . (Note that $f(\mathbf{0}) = 0$, which is the minimum of f over the unit disk when $\mu > 0$).

Section 5.2 (p. 340)

4. The required volume is

$$\int_0^1 \int_0^1 \int_0^{x^2+y^4} dz dy dx = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}.$$