Homework answers, week of Feb. 10, 2009

Review Exercises, Chapter 2 (p. 176)

23. Fix $\mathbf{x}$. Then $g(\lambda) = f(\lambda\mathbf{x})$ is a function of a single variable, namely λ. The chain rule implies

$$\frac{d}{d\lambda}g(\lambda) = \frac{d}{d\lambda}f(\lambda\mathbf{x}) = \nabla f(\lambda\mathbf{x}) \cdot \frac{d}{d\lambda}(\lambda x) = \nabla f(\lambda\mathbf{x}) \cdot \mathbf{x}.$$

When $\lambda = 1$, we have $g'(1) = \nabla f(\mathbf{x}) \cdot \mathbf{x}$. On the other hand, since f is homogeneous of degree p, we have

$$f(\lambda\mathbf{x}) = \lambda^p f(\mathbf{x}),$$

so if $\mathbf{x}$ is fixed, then

$$\frac{d}{d\lambda}f(\lambda\mathbf{x}) = \frac{d}{d\lambda}\lambda^p f(\mathbf{x}) = p\lambda^{p-1}f(\mathbf{x}).$$

When $\lambda = 1$, this is just $pf(\mathbf{x})$, which proves the result.

Section 3.1 (p. 192)

19. A straightforward calculation shows that

$$\frac{\partial^2 u}{\partial x^2} = 6x \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -6x,$$

which gives the result.

21. Use the chain rule. It might be helpful to regard f as a function of a single variable, say $f = f(y)$, and y as a function of two variables: $y = y(x,t) = x - t$. We have

$$\frac{\partial}{\partial x}f(x-t) = \frac{df}{dy}\frac{dy}{dx} = f'(x-t)$$

and

$$\frac{\partial}{\partial t}f(x-t) = \frac{df}{dy}\frac{dy}{dt} = -f'(x-t).$$

Similar calculations give $(\partial/\partial x)g(x+t) = g'(x+t)$ and $(\partial/\partial t)g(x+t) = g'(x+t)$. The second derivatives are the derivatives of the first derivatives:

$$\frac{\partial}{\partial x}f'(x-t) = \frac{df'}{dy}\frac{dy}{dx} = f''(x-t)$$

and

$$\frac{\partial}{\partial t} - f'(x-t) = \frac{df'}{dy}\frac{dy}{dt} = +f''(x-t).$$

Similarly, $(\partial^2/\partial x^2)g(x+t) = g''(x+t)$ and $(\partial^2/\partial t^2)g(x+t) = g''(x+t)$. Since

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 g}{\partial x^2},$$

and similarly for $\partial^2\phi/\partial t^2$, the result follows.

Section 3.3 (p. 223)

22. One approach is that in Example 8. Each point on the plane has the form

$$\left(x, y, \tfrac{1}{2}(20 - 2x + y)\right).$$

The square of the distance of such a point from the origin is

$$f(x,y) = x^2 + y^2 + (10 - x + \tfrac{1}{2}y)^2,$$

which is minimized when $\nabla f(x,y) = (0,0)$, that is,

$$\begin{aligned} 0 &= 4x - y - 20 \\ 0 &= \tfrac{5}{2}y - x + 10. \end{aligned}$$

The solution is $x = 40/9$, $y = -20/9$, which implies $f_{\min} = 400/9$. Since f is the square of the distance, the actual distance of the closest point to the origin is $20/3$.