

Homework for week of Feb. 2

From pages 159–161.

8. Use the chain rule, where all derivatives are evaluated at $\mathbf{x} = (\rho, \theta, \phi)$:

$$\frac{\partial f}{\partial \rho} = \nabla f \cdot (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$$

$$\frac{\partial f}{\partial \theta} = \nabla f \cdot (-\rho \sin \theta \sin \phi, \rho \cos \theta \sin \phi, 0)$$

$$\frac{\partial f}{\partial \phi} = \nabla f \cdot (\rho \cos \theta \cos \phi, \rho \sin \theta \cos \phi, -\rho \sin \phi)$$

which yields

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x} \cos \theta \sin \phi + \frac{\partial f}{\partial y} \sin \theta \sin \phi + \frac{\partial f}{\partial z} \cos \phi$$

$$\frac{\partial f}{\partial \theta} = -\frac{\partial f}{\partial x} \rho \sin \theta \sin \phi + \frac{\partial f}{\partial y} \rho \cos \theta \sin \phi$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x} \rho \cos \theta \cos \phi + \frac{\partial f}{\partial y} \rho \sin \theta \cos \phi - \frac{\partial f}{\partial z} \rho \sin \phi$$

12. There are two equivalent ways to proceed: use the chain rule or simply substitute the path at the start and take ordinary derivatives. The latter is probably simpler in this case.

(a) Substitution yields $T(\mathbf{c}(t)) = t^2$, so $T'(t) = 2t$.

(b) Use the usual linearization:

$$\begin{aligned} T(\pi/2 + 0.01) &\approx T(\pi/2) + T'(\pi/2) \cdot 0.01 \\ &= 1 + (\pi/2)^2 + 2(\pi/2 + 0.01) \\ &= 1.02 + (\pi/2)^2 + \pi \\ &\approx 6.63. \end{aligned}$$

18. Given $F(x(y,z), x(y,z), z(x,y)) = \text{constant}$, the chain rule implies that

$$\frac{\partial F}{\partial x} = \nabla F(x, y, z) \cdot \left(1, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right),$$

and if we hold z constant, then $\partial z/\partial x = 0$, so

$$\frac{\partial y}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial y},$$

assuming that the denominator is not zero. Similar computations imply that

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

when x is held fixed and

$$\frac{\partial x}{\partial z} = -\frac{\partial F/\partial z}{\partial F/\partial x}$$

when y is held fixed. Hence

$$\left(\frac{\partial y}{\partial x}\right) \left(\frac{\partial z}{\partial y}\right) \left(\frac{\partial x}{\partial z}\right) = \left(-\frac{\partial F/\partial x}{\partial F/\partial y}\right) \left(-\frac{\partial F/\partial y}{\partial F/\partial z}\right) \left(-\frac{\partial F/\partial z}{\partial F/\partial x}\right) = -1.$$

28. The function f cannot be differentiable if f is not defined and continuous at $x = 0$. If $p = 0$, then $f(x) = \sin(1/x)$ when $x \neq 0$, but f does not have a limit as $x \rightarrow 0$. Similarly, if $p < 0$, then x^p does not tend to 0, so again f has no limit as $x \rightarrow 0$. Therefore we require $p > 0$. If $x > 0$, then the chain and product rules imply

$$\frac{df}{dx} = px^{p-1} \sin\left(\frac{1}{x}\right) - x^{p-2} \cos\left(\frac{1}{x}\right),$$

which is not continuous at $x = 0$ if $p \leq 2$.