Answers to in-class discussion problems

We have two surfaces: S_1 is the cylindrical shell centered about the z axis of radius 2 and height 1 ($0 \leq z \leq 1$), and S_2 is the half of the sphere of radius 2 above the xy plane.

1. Compute $\iint_{S_1} z\,dS$.

   Solution: Parametrize S_1 as $\Phi(\theta, z) = (2\cos\theta, 2\sin\theta, z)$. Then a calculation shows

   $$dS = \|\mathbf{T}_\theta \times \mathbf{T}_z\|\,d\theta\,dz = 2\,d\theta\,dz$$

   so

   $$\iint_{S_1} z\,dS = \int_0^1 \int_0^{2\pi} 2z\,d\theta\,dz = 2\pi.$$

2. Compute $\iint_{S_2} z\,dS$.

   Solution: Using the usual spherical coordinates, we find

   $$dS = \|\mathbf{T}_\theta \times \mathbf{T}_\phi\|\,d\phi\,d\theta = 4\sin\phi\,d\phi\,d\theta$$

   (see the April 6 lecture notes). Therefore,

   $$\iint_{S_2} z\,dS = \int_0^{2\pi} \int_0^{\pi/2} (2\cos\phi)(4\sin\phi)\,d\phi\,d\theta = 8\pi.$$

3. Compute $\int_C z\,ds$ where C is an arc extending from the equator of the sphere in Problem 2, going across the north pole, then back to the equator 180° opposite.

   Solution: Since the integrand is independent of x and y, we may use any convenient longitude line. Suppose we take the one in the yz plane, parametrized as $\mathbf{r}(t) = (0, 2\cos t, 2\sin t)$ for $0 \leq t \leq \pi$. Then $ds = \|\mathbf{r}'(t)\|\,dt = 2\,dt$ so that

   $$\int_C z\,ds = \int_0^\pi (2\sin t)\cdot 2\,dt = 8.$$

4. Compute $\iint_{S_1} z\mathbf{k}\cdot d\mathbf{S}$.

   Solution 1: Since a normal to S_1 is $(x, y, 0)$, the vector field $z\mathbf{k}$ is perpendicular to S_1 at every point, so the flux is 0.

Solution 2: Parametrize S_1 as $\Phi(\theta,z) = (2\cos\theta, 2\sin\theta, z)$. Then a normal vector to S_1 is

$$\mathbf{T}_\theta \times \mathbf{T}_z = (2\cos\theta, 2\sin\theta, 0)$$

so that

$$\iint_{S_1} z\mathbf{k}\cdot d\mathbf{S} = \int_0^1 \int_0^{2\pi} z\mathbf{k}\cdot(2\cos\theta, 2\sin\theta, 0)\,d\theta\,dz = 0.$$

5. Compute $\iint_{S_2} z\mathbf{k}\cdot d\mathbf{S}$.

   Solution: The usual spherical coordinates give an inward pointing normal, so negate it to get its outward-pointing counterpart. This gives (see the April 6 lecture notes):

   $$\mathbf{T}_\theta \times \mathbf{T}_\phi = 4\sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

   Therefore, $z\mathbf{k}\cdot(\mathbf{T}_\theta \times \mathbf{T}_\phi) = 4\sin\phi\cos\phi$, so

   $$\iint_{S_2} z\mathbf{k}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi/2} 4\sin\phi\cos\phi\,d\phi\,d\theta = 4\pi.$$

6. Compute $\iint_C z\mathbf{k}\cdot d\mathbf{r}$, where C is any longitude line as in Problem 3.

   Solution 1: If $\mathbf{T}$ is the unit tangent to C at any point, then $z\mathbf{k}\cdot\mathbf{T} > 0$ as we head northward to the pole and $z\mathbf{k}\cdot\mathbf{T} < 0$ as we head south on the opposite side. By symmetry, the two quantities must cancel, so

   $$\iint_C z\mathbf{k}\cdot d\mathbf{r} = 0.$$

   Solution 2: Choose C as in Problem 3. This gives

   $$\int_C z\mathbf{k}\cdot d\mathbf{r} = \int_0^\pi (2\sin t)\mathbf{k}\cdot(0, -2\sin t, 2\cos t)\,dt = \int_0^\pi 4\sin t\cos t\,dt = 0.$$

7. Let $\mathbf{F} = \mathbf{r}/\|\mathbf{r}\|$ and find $\iint_{S_1}\mathbf{F}\cdot d\mathbf{S}$ assuming an outward pointing normal.

   Solution 1: Parametrize S_1 as $\Phi(\theta,z) = (2\cos\theta, 2\sin\theta, z)$. Then a normal vector to S_1 is

   $$\mathbf{T}_\theta \times \mathbf{T}_z = (2\cos\theta, 2\sin\theta, 0)$$

so that

$$\begin{aligned}\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} &= \iint_{S_1} \mathbf{F}\cdot(\mathbf{T}_\theta \times \mathbf{T}_z)\, d\theta\, dz \\ &= \int_0^1 \int_0^{2\pi} \frac{4}{\sqrt{4+z^2}}\, d\theta\, dz \\ &= 8\pi \sinh^{-1}\left(\frac{1}{2}\right).\end{aligned}$$

Solution 2: We have the normal vector $(x,y,0)$—this is not a unit normal but is equivalent to $\mathbf{T}_\theta \times \mathbf{T}_z$ as above. Thus $\mathbf{F}\cdot d\mathbf{S} = (x^2+y^2)/\sqrt{x^2+y^2+z^2} = 4/\sqrt{4+z^2}$ since the sides lie over the circle $x^2+y^2=2^2$. Therefore,

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \int_0^1 \int_0^{2\pi} \frac{4}{\sqrt{4+z^2}}\, d\theta\, dz$$

as above.

8. Let $\mathbf{F} = \mathbf{r}/\|\mathbf{r}\|$ and find $\iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$, with an outward pointing normal.

   Solution 1: The usual spherical coordinates give an inward pointing normal, so negate it to get its outward-pointing counterpart. This gives (see the April 6 lecture notes):

   $$\mathbf{T}_\theta \times \mathbf{T}_\phi = 4\sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

   After some simplification, we find

   $$\mathbf{F}\cdot(\mathbf{T}_\theta \times \mathbf{T}_\phi) = 4\sin\phi,$$

   so that

   $$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi/2} 4\sin\phi\, d\phi\, d\theta = 8\pi.$$

   Solution 2: A *unit* normal vector $\mathbf{n}$ to S_2 is $\frac{1}{2}(x,y,z)$, so that $\mathbf{F}\cdot\mathbf{n} = (x^2+y^2+z^2)/2\sqrt{x^2+y^2+z^2} = 1$. Thus

   $$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_2} 1\, dS,$$

   which is just the surface area of the hemisphere, i.e., $2\pi\cdot 2^2 = 8\pi$.