Does synchronization of networks of chaotic maps lead to control?

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Abstract

We consider networks of chaotic maps with different network topologies. In each case, they are coupled in such a way as to generate synchronized chaotic solutions. By using the methods of control of chaos we are controlling a single map into a predetermined trajectory. We analyze the reaction of the network to such a control. Specifically we show that a line of 1-d logistic maps that are unidirectionally coupled can be controlled from the first oscillator whereas a ring of diffusively coupled maps cannot be controlled for more than 5 maps. We show that rings with more elements can be controlled if every third map is controlled. The dependence of unidirectionally coupled maps on noise is studied. The noise level leads to a finite synchronization lengths for which maps can be controlled by a single location. A 2-d lattice is also studied.

It is well known that strongly coupled chaotic oscillators may synchronize and oscillate collectively in a chaotic way. Additionally, due to the inherent instability of chaotic systems, we can steer trajectories in a desired direction using small control efforts. Given a network of synchronized chaotic maps, this study determines whether it is possible to control such a network from one point in the network using information just from a neighborhood of this point. We find that the topology of the network as well as the existence of noise in the network plays a significant role in determining the size of the spatial correlation that allows control.

I. INTRODUCTION

Control of chaos ([6]) and synchronization of chaos [7] are well understood phenomena associated with chaotic oscillators. This study considers the following question: Assuming a synchronized network of chaotic oscillators, can we control the network locally into a pre-specified trajectory and use the synchronization to force the rest of the network to follow the local control. We will study how such forced synchronization depends on the network topology, the coupling parameters, the inherent parameters of the chaotic oscillators and on noise.

A. Controlling Chaos

In the following we will focus on maps - representing Poincare maps or time sampled dynamical systems. We follow closely the discussion in [9] which is restricted to one dimensional maps or periodic orbits of dynamical systems with a one dimensional unstable manifold. For control in higher dimensions see [1]. Let

$$Z_{i+1} = F(Z_i, p), \tag{1}$$

where $Z_i \in \mathbf{R}^n, p \in \mathbf{R}$ and F is sufficiently smooth in both variables. Here, p is considered a real parameter which is available for external adjustment but is restricted to lie in some small interval,

$$\mid p - \bar{p} \mid < \delta \tag{2}$$

around a nominal value $\bar{p}$. If we want to stabilize an unstable fixed point $Z_*(\bar{p})$ on the attractor by a linear feedback

$$p - \bar{p} = -K^T[Z_i - Z_*(\bar{p})] \tag{3}$$

we have to determine the n-dimensional column vector K in such a way that

$$Z_{i+1} - Z_*(\bar{p}) = (A - BK^T)[Z_i - Z_*(\bar{p})] \tag{4}$$

has a stable fixed point, i.e. all eigenvalues of the matrix $A - BK^T$ have modulus smaller than one. Here A is the $n \times n$ Jacobian matrix and B is an n-dimensional column vector,

$$A = D_Z F(Z_*(\bar{p}), \bar{p}), \tag{5}$$

$$B = D_p F(Z_*(\bar{p}), \bar{p}), \tag{6}$$

This can be done by pole-placement or other methods [9]. Extensions to control period N orbits are done by controlling fixed points of the N-fold composite maps.

B. Synchronization of Chaos

The general theory of synchronization of chaotic oscillators is presented in ([7], Chapter 14). We present here only the main definitions and results necessary for our study. Let f denote a 1-d non-linear map. We consider M identical coupled chaotic maps subject to linear coupling. We represent the coupling scheme by means of a general linear operator L given by an $M \times M$ matrix:

$$x_i^{n+1} = \sum_{j=1}^{M} L_{ij} f(x_j^n) \tag{7}$$

We require that the following properties hold for the coupling operator:

- The system (7) has a symmetric synchronous solution where all states are identical. This is true if the constant vector $\mathbf{e_1} = (\mathbf{1}...\mathbf{1})$ is an eigenvector of the matrix L to the eigenvalue $\sigma_1 = 1$.
- All other eigenvalues of L are in absolute value less than 1.

Linear stability theory for the synchronous state determines its stability: With $\sigma_1 = 1$ the largest eigenvalue of L, the largest asymmetric perturbation is determined by σ_2 leading to a transverse Lyapunov exponent $\lambda_\perp = \lambda + \ln|\sigma_2|$ with λ the Lyapunov exponent of f.

Hence stability of the synchronous state is characterized by

$$ln|\sigma_1| - ln|\sigma_2| > \lambda \tag{8}$$

requiring a gap (of the size at least λ) in the spectrum of the linear coupling operator L between the first and the second eigenvalues. A lot of work has been done to determine the relationship between the connectivity matrix L, the topology of the connections and the synchronization properties of the associated map lattices [2–4]. Extensions to synchronized behavior of coupled dynamical systems have been discussed by Pogromsky and Nijmeijer [8].

To keep things simple and to work out the main ideas we will consider here only two typical 1-D coupling schemes:

- **Unidirectional coupling** implies that the signal from one chaotic oscillator forces the next one. In this case the coupling matrix has the form

$$L = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \epsilon & 1-\epsilon & 0 & \dots & 0 \\ 0 & \epsilon & 1-\epsilon & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \epsilon & 1-\epsilon \end{bmatrix}.$$

  We impose free boundary condition on the first oscillator.

- **Nearest Neighbor Coupling/Diffusive Coupling** A coupling matrix of the form

$$L = \begin{bmatrix} 1-2\epsilon & \epsilon & 0 & \dots & \epsilon \\ \epsilon & 1-2\epsilon & \epsilon & \dots & 0 \\ 0 & \epsilon & 1-2\epsilon & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \epsilon & \dots & 0 & \epsilon & 1-2\epsilon \end{bmatrix}$$

  represents nearest neighbor or diffusive coupling. Typically, periodic boundary conditions are imposed such that the topology becomes that of a ring of coupled oscillators.

II. CONTROL OF NETWORKS OF MAPS

A. Unidirectional coupling

We are trying to control a line of identical logistic maps $x^{n+1} = px^n(1-x^n)$ by stabilizing the first map on an unstable fixed point through variations in the parameter p leading to a time dependent sequence of control states denoted by p^n. In a unidirectional coupling system, the first oscillator is decoupled from the others. Hence our system is represented by

$$\vec{X}^{n+1} = F(\vec{X}^n, p^n) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \epsilon & 1-\epsilon & 0 & \dots & 0 \\ 0 & \epsilon & 1-\epsilon & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \epsilon & 1-\epsilon \end{bmatrix} \begin{bmatrix} f(x_1^n, p^n) \\ f(x_2^n, \bar{p}) \\ f(x_3^n, \bar{p}) \\ \vdots \\ f(x_M^n, \bar{p}). \end{bmatrix} \tag{9}$$

We define $\vec{X}^* = x^*[1, 1, \dots, 1]^T$ to be the vector of fixed points and linearize eq.(9) in a neighborhood of $\vec{X}^*$ and $\bar{p}$.

$$\vec{X}^{n+1} - \vec{X}^* \approx AL(\vec{X}^n - \vec{X}^*) + B(p^n - \bar{p})[1, 0, \dots, 0]^T \tag{10}$$

with A and B given as

$$A = \frac{df}{dx} = \bar{p} - 2\bar{p}x^* \tag{11}$$

$$B = \frac{df}{dp} = x^*(1-x^*) \tag{12}$$

Since the first map has no feedback from any other map, we can control the eigenvalue of the first map at its fixed point independently of all the others. Calling that eigenvalue λ_1 we can therefore assume that $|\lambda_1| < 1$. Then the linearization becomes

$$\vec{X}^{n+1} - \vec{X}^* = A\tilde{L}(\vec{X}^n - \vec{X}^*) \tag{13}$$

with $\tilde{L}$ being the form

$$\tilde{L} = \begin{bmatrix} \frac{\lambda_1}{A} & 0 & 0 & \dots & 0 \\ \epsilon & 1-\epsilon & 0 & \dots & 0 \\ 0 & \epsilon & 1-\epsilon & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \epsilon & 1-\epsilon \end{bmatrix}$$

Therefore stability of whole unidirectional chain of maps on the fixed point $x^* = 1 - \frac{1}{\bar{p}}$ is given by

$$|(1-\epsilon)\frac{df}{dx}| = |(1-\epsilon)(2-\bar{p})| < 1. \tag{14}$$

This gives us a relationship between synchronization strength ϵ and the control parameter p: For

$$\bar{p} < 2 + \frac{1}{1-\epsilon} \tag{15}$$

the complete line of unidirectionally coupled maps can be stabilized. Note that the left hand side of Equation (14) is independent of the actual map. Hence for any arbitrary unidirectional map the complete line of coupled maps can be stabilized, if there exists a critical parameter value p^* such that for $p < p^*$ or $p > p*$, $|(1-\epsilon)\frac{df}{dx}(p^*)| < 1$.

We can also easily control nontrivial periodic orbits (period greater than one). In that case, the requirement of stability is basically the same as Eq.(14) replacing $\frac{df}{dx}$ by the average of the derivatives evaluated over all the points in the periodic orbit.

B. Diffusive coupling

We intend to control a ring of 1-dimensional diffusively coupled maps by controlling just one map to its unstable fixed point. The linearization near the fixed point X^* and the parameter value $\bar{p}$ becomes

$$\vec{X}^{n+1} - \vec{X}^* = AL(\vec{X^n} - \vec{X^*}) + (1-2\epsilon)B(p^n - \bar{p})[1, 0, \ldots, 0]^T \tag{16}$$

with

$$A = \frac{df}{dx} = \bar{p} - 2\bar{p}x^* \tag{17}$$

$$B = \frac{df}{dp} = x^*(1-x^*) \tag{18}$$

and L being the ring coupling matrix given in Section (I B). The linear approximation for the first oscillator is:

$$x_1^{n+1} - x^* = A((1-2\epsilon)(x_1^n - x^*) + \epsilon(x_2^n - x^*)) + \epsilon(x_M^n - x^*) + B(1-2\epsilon)(p^n - \bar{p}) \tag{19}$$

Choosing p^n such that the right hand side of Eq.(19) is zero, trivially stabilizes the first oscillator and decouples it from the rest of the system. The associated feedback control is

$$\begin{aligned} p^n - \bar{p} &= -\tfrac{A}{B}(x_1^n - x^*) \\ &- \tfrac{A\epsilon}{B(1-2\epsilon)}((x_2^n - x^*) + (x_M^n - x^*)) \end{aligned} \tag{20}$$

giving us a linearization of

$$\vec{X}^{n+1} - \vec{X}^* = A\tilde{L}(\vec{X}^n - \vec{X}^*) \tag{21}$$

with $(\tilde{L})$ being

$$\tilde{L} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \epsilon & 1-2\epsilon & \epsilon & \dots & 0 \\ 0 & \epsilon & 1-2\epsilon & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \epsilon & \dots & 0 & \epsilon & 1-2\epsilon \end{bmatrix}. \tag{22}$$

Thus the stability of our control algorithm depends on the eigenvalues of the matrix $A\tilde{L}$ which are 0 and the eigenvalues of a $(M-1)\times(M-1)$ tridiagonal matrix

$$T = \begin{bmatrix} 1-2\epsilon & \epsilon & 0 & \dots & 0 \\ \epsilon & 1-2\epsilon & \epsilon & \dots & 0 \\ 0 & \epsilon & 1-2\epsilon & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \epsilon & 1-2\epsilon \end{bmatrix}. \tag{23}$$

The eigenvalues of T are $\lambda_j = 1 - 2\epsilon(1 - cos\frac{\pi j}{M})$. Hence the condition of stability of our control algorithm is

$$|A[1 - 2\epsilon(1 - cos\frac{\pi j}{M})]| < 1 \qquad \forall j = 1, \dots, M-1. \tag{24}$$

Since the left hand side of this expression is linear in $cos\frac{\pi j}{M}$, the extrema can only occur at $j = 1$ or $M - 1$ for every fixed ϵ. Thus the above inequality is equivalent to

$$\begin{cases} |1 - 2\epsilon(1 - cos\frac{\pi}{M})| < \frac{1}{|A|} \\ |1 - 2\epsilon(1 - cos\frac{(M-1)\pi}{M})| < \frac{1}{|A|}. \end{cases} \tag{25}$$

Since $1 - 2\epsilon(1 - cos\frac{\pi}{M}) > 1 - 2\epsilon(1 - cos\frac{(M-1)\pi}{M})$ (for $\epsilon > 0$), it is also equivalent to

$$\begin{cases} 1 - 2\epsilon(1 - cos\frac{\pi}{M}) < \frac{1}{|A|} \\ 1 - 2\epsilon(1 - cos\frac{(M-1)\pi}{M}) > -\frac{1}{|A|} \end{cases} \tag{26}$$

which gives the stability condition

$$\frac{|A| - 1}{2|A|(1 - cos\frac{\pi}{M})} < \epsilon < \frac{|A| + 1}{2|A|(1 + cos\frac{\pi}{M})}. \tag{27}$$

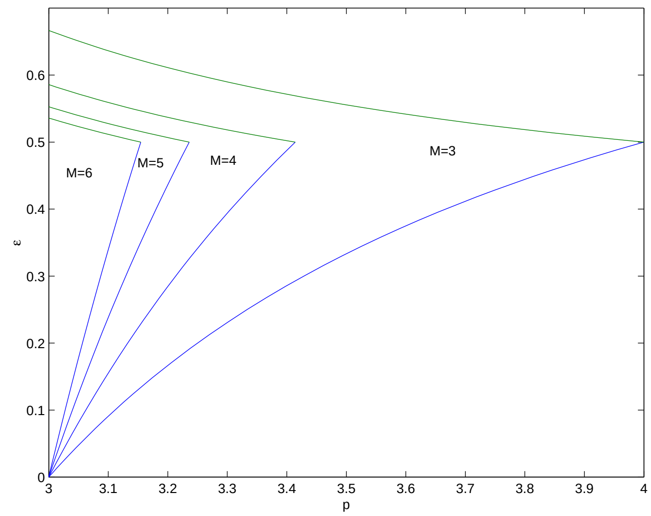

FIG. 1: Controllable regime of ϵ for different $\bar{p}$ and M

For such an ϵ to exist, we need

$$cos\frac{\pi}{M} < \frac{1}{|A|.} \tag{28}$$

This condition is violated for $M > 3$. Using the fact that $A = 2 - \bar{p}$ for a control on a fixed point we can analytically determine from Equations (27) and (28) the relationship between p, the number of oscillators in the ring M and the strength of the coupling ϵ. The results are shown in in figure 1. We see that in the chaotic regime $p > 3.566...$ only a ring of three maps can be controlled.

Remarks:

- Notice that Equation (28) does not depend on the particulars of the logistic map. As long as we can decouple the controlled oscillator from all the others by choosing p in Equation (19) in such a way that the eigenvalue of the first oscillator becomes zero, the condition in Equation (28) holds.

- The control algorithm depends critically on A (see Eq.28). If $|A|$ is not very large, say it is just a bit lager than 1, then $\frac{1}{A}$ is close to one and we are able to control a system with a large number of maps. This implies that we should be able to control a large ring of maps to an unstable periodic orbit that has just lost stability through a period doubling bifurcation. For example, for $\bar{p} = 3.843$ we get $|A| \approx 1.0381$ for the period-3 orbit just after the main stable period three window. By eq.(28), we should be able to control a system with up to 11 oscillators. Our numerical simulations allow us to control 8 maps for as long as we choose to simulate - rings with 9 and 10 maps eventually lose stability due to what we believe is an accumulation of numerical errors.

- Our feedback policy (Equation 20) is conservative by setting the eigenvalue for the first map to zero. By choosing different feedback coefficients we can increase the first eigenvalue to values that are still stable but just below 1. At that moment Equation (28) does not hold any more. However, we can then decrease the other eigenvalues and stabilize more than just 3 oscillators. The operating principle in that case is that the feedback has to introduce strong asymmetry in order to change the symmetry type of the ring of oscillators from a diffusive coupling to a preferred direction coupled ring. It is well known, that in such a case, synchronization is much easier to achieve ([7]). In these cases the matrix $\tilde{L}$ does not decouple any more and we do not have analytical eigenvalues any more. By trial and error we found that we can stabilize rings of up to five coupled maps with a feedback just to the first oscillator based on the state of this oscillator and the two adjacent oscillators.

- A ring of a large number of oscillators can be controlled via periodic refreshers. For instance, Figure 2 shows the third and the ninth map of a twelve map ring. We choose $\bar{p} = 3.83$ and a coupling parameter $\epsilon = 0.48$ and control every third map by a local feedback control like Equation 20. The control is started at around $n = 5500$. We see that the two very distant maps go from synchronized chaos to the same stabilized fixed point.

- We can extend this scheme to control a two dimensional lattice of identical maps(oscillators) coupled diffusively to its nearest neighbors. Suppose x_{ij} denote the oscillator on the ith row and jth column, then

$$x_{ij}^{n+1} = (1-4\epsilon)f(x_{ij}^n) + \epsilon[f(x_{i-1j}^n) + f(x_{i+1j}^n) + f(x_{ij-1}^n) + f(x_{ij+1}^n)]. \tag{29}$$

  Suppose we are going to control a single oscillator x_{ij} in the lattice. By the same analysis we have done in one-dimensional coupling case, we get the control algorithm

$$\begin{aligned} p_{ij}^n - \bar{p} \; &= \; -\tfrac{A}{B}(x_{ij}^n - x^*) - \tfrac{\epsilon A}{(1-4\epsilon)B}[(x_{i-1j}^n - x^*) \\ &+ \; (x_{i+1j}^n - x^*) + (x_{ij-1}^n - x^*) + (x_{ij+1}^n - x^*)] \end{aligned} \tag{30}$$

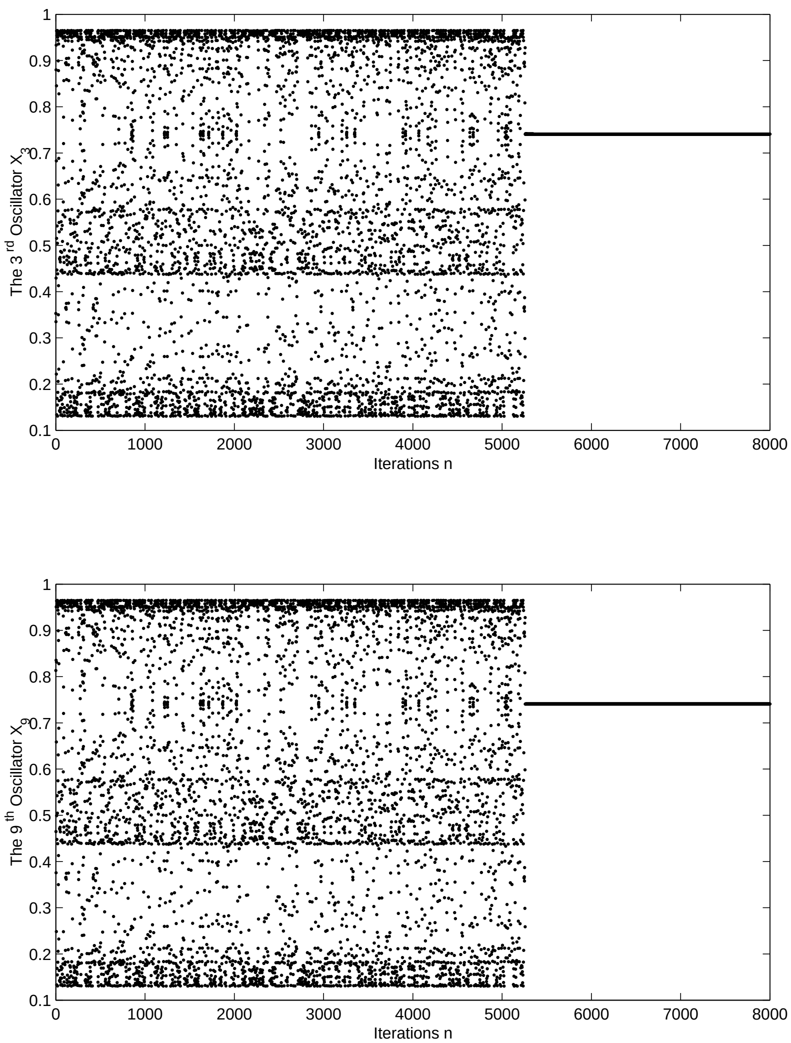

FIG. 2: Control of a ring of logistic maps with local feedback control at every third map.

Numerical simulation shows that we cannot control the whole system by only controlling one single oscillator. However, we can control the lattice by a refresher lattice. We find that by controlling the maps locally in the odd number columns in the odd rows and all the maps in the even number columns in the even rows we can control the whole lattice to a fixed point.

- There are no fundamental obstacles to extending these results to the control of periodic orbits of continuous time oscillators as long as the periodic orbit has a 1 dimensional unstable manifold. The resulting Poincare map transforms a continous system back to the case of coupled maps.

III. THE IMPACT OF NOISE ON CONTROLLING A UNIDIRECTIONAL LINE OF MAPS.

We add independent identically distributed random noise ξ_i to the coupled maps chosen from a uniform distribution on the interval $[-c, c]$. We use c as a measure of the magnitude of the noise. Then the system becomes

$$\vec{X}^{n+1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \epsilon & 1-\epsilon & 0 & \dots & 0 \\ 0 & \epsilon & 1-\epsilon & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \epsilon & 1-\epsilon \end{bmatrix} \begin{bmatrix} f(x_1^n, p^n) \\ f(x_2^n, \bar{p}) \\ f(x_3^n, \bar{p}) \\ \vdots \\ f(x_M^n, \bar{p}) \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \\ \xi_n \end{bmatrix} \tag{31}$$

While the theory shows that for any finite number of maps, the control algorithm leads to stable fixed points, we find that as the noise level increases, maps downstream decouple and cannot be controlled. Specifically, let

$$\eta_i = \max_{n \in Q} \mid x_i^n - x^* \mid \quad \textit{and} \quad \gamma_i = \frac{\eta_i}{c},$$

where η_i describes the maximal displacement of the map i from its fixed point over the whole controlled set of time steps Q and γ_i is the gain over the input noise level.

Performing a numerical experiment over a few thousand iterations where we control a uni-directional lattice with length 20 and added noise of size 10^{-4} we get Figure (3). It shows that γ_i grows exponentially as a function of the map index. Throughout our simulations, the exponential growth model only varies in a small range for different cases. If we set $\gamma_i = ae^{ib}$, we numerically observe $a \in (0.75, 0.85)$ and $b \in (0.401, 0.408)$. This offers a way to predict the noise growth in the system. Choosing a criterion of having control of the Mth oscillator by requiring $\eta_M = c\gamma_i = cae^{Mb} \leq \epsilon$ for some small constant ϵ ($\epsilon = 10^{-4}$ in our simulation) we calculate that for

$$M < \frac{\ln \frac{\epsilon}{ca}}{b} \tag{32}$$

the unidirectional chain of maps that can be controlled from the first map (see Figure 4). Again, putting refresher controls after M maps allows us to control an unlimited chain. Figure 5 shows the exponential growth along a chain and the periodic resets due to refresher control. The added noise had an amplitude of 10^{-7} and the maximal allowable perturbation

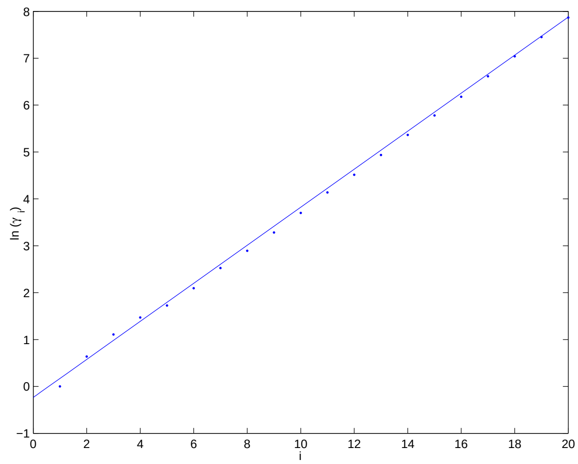

FIG. 3: Noise propagation in the unidirectional coupled system 31. Numerical data points are fitted with an exponential model.

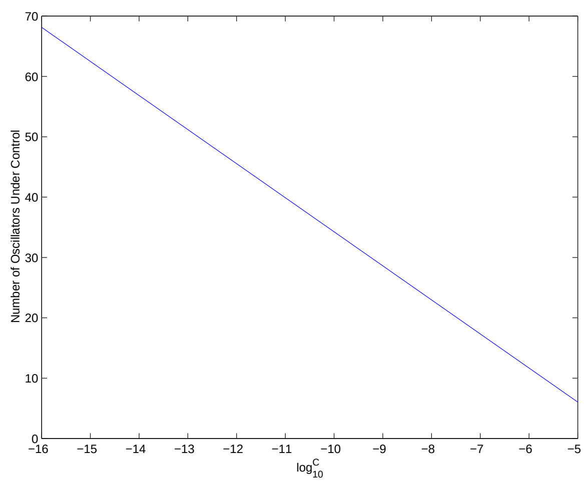

FIG. 4: Number of Oscillators Under Control For Different Input Noise Level (numerical experiment).

was set at ($\epsilon = 10^{-4}$). With the exponential growth along the chain from above this indicates that the 18th map will exceed the allowable noise level and hence should be controlled back to zero.

This result is closely related to earlier results of Kaneko [5] who showed that a unidirectional chain of synchronized logistic maps can become convectively unstable, i.e perturbations will travel downstream, amplify and generate spatio-temporal chaos. Our analysis shows that (i) in the absence of extra noise, the convective instability can be suppressed;

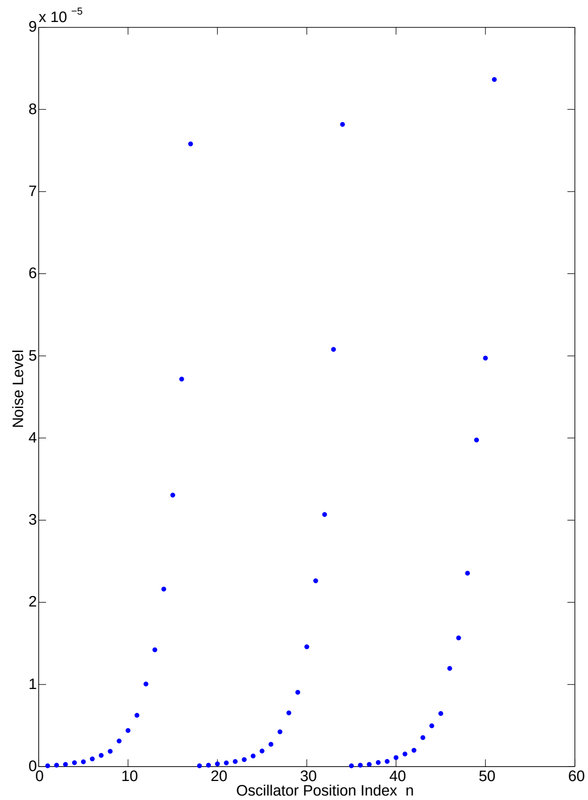

FIG. 5: Numerical simulation of a unidirectional chain with added noise level of 10^{-7}. Shown is the growth of the perturbation from steady state as a function of the map number. Every 18th map is controlled to zero.

(ii) even with strong control at the leading map of the unidirectional chain, with added noise the noise will exponentially grow in space, (iii) with suitably placed control action, the convective instability mode will stay below detection levels.

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[1] D. Auerbach, C. Grebogi, E. Ott and J. Yorke: Controlling chaos in high dimensional systems, Phys. Rev. Lett. **69**(24), 3481 (1992)

[2] Y.Chen, G. Rangaranjan, M.Ding, Generalized stability analysis of synchronized dynamics in coupled systems, Phys. Rev. E. **67**, 026209 (2003)

[3] P.M. Gade, R. Amritkar, Spatially periodic orbits in coupled-map lattices, Phys. Rev. E **47** 143 (1993)

[4] J. Jost, M.P.Joy, Spectral properties and synchronization in coupled map lattices, Phys. Rev. E 65, 016201 (2001)

[5] Kunihiko Kaneko, Lyapunov analysis and information flow in coupled map lattices. Physica D **23** pp 436 - 447 (1986)

[6] E. Ott, C.Grebogi, J.A. Yorke: Controlling Chaos, Phys. Rev. Lett. **64** 1196 (1990)

[7] A.S. Pikovsky, M.G. Rosenblum and J. Kurths: *Synchronization - A Unified Approach to Nonlinear Science* Cambridge University Press, (2001)

[8] A. Yu Pogromsky, H. Nijmeijer, Cooperative oscillatory behavior of mutually coupled dynamical systems, IEEE Trans. Circuits Syst. I:Fundamental Theory and Applications **48**(2), 1057 (2001)

[9] E. Romeiras, C. Grebogi, E. Ott, W.P. Dayawansa: Controlling chaotic dynamical systems, Physica D **58**, 165 (1992)