Bucket Brigades when Worker Speeds do not Dominate Each Other Uniformly

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Bucket Brigades when Worker Speeds do not Dominate Each Other Uniformly*

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Abstract

A two worker bucket brigade is studied where one worker has a constant speed over the whole production line and the other is slower over the first portion and faster over the second portion of the line. We analyze the dynamics and throughput of the bucket brigade under two different assumptions: (i) workers can pass each other, and (ii) workers are blocked when an upstream worker runs into a downstream worker. We show that a slight modification of the bucket brigade will always lead to a self organizing production line. The bucket brigade either balances to a fixed point or settles into a period-two orbit. Insights for the management of the bucket brigades for the various scenarios are discussed using results on the throughput performance and self-organization. Extensions to multiple skill levels and more workers are outlined.

1 Introduction

The widespread use of cross-training in industrial settings has led to a growing interest to study operational control in systems with multi-skilled, or flexible workers. In addition to papers that study optimal control of work sharing and worker assignment, there exists considerable work on policies that are defined by a set of rules that tells each worker what to do "next".

The Toyota Sewn Products Management System (TSS) is one of the most widely studied architectures of this kind. TSS is employed regularly by manufacturers of sewn products in modules that are used in the finishing and assembly of cut parts into a subassembly or finished garment. The production modules are typically U-shaped, and workers process garments as a team. In a TSS line, each worker picks up a task and processes (also carries) it at each station until he gets bumped by a downstream worker. There is no additional WIP buffers kept. The ordering of the workers has to be preserved but other than that, workers are not restricted to any particular zones. When a worker comes to a busy station he must wait until the station becomes available; he also may not seek other work. The number of machines in a TSS line typically ranges from 2 to 16 with an average of 2.5 machines per worker (Bischak, 1996). In addition to apparel and garment industry, TSS lines have also been shown to perform robustly in certain types of warehousing environments (Bartholdi and Eisenstein, 1996).

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The term "bucket brigade" was coined by Bartholdi and Eisenstein (1996) for TSS lines in which the workers are sequenced from slowest to fastest. The authors provided the first comprehensive analysis of the dynamics of such systems and showed that the bucket brigade is self-balancing; that is, eventually a stable partition of work among workers will emerge such that each worker repeatedly executes the same interval of work content. The main result in Bartholdi and Eisenstein (1996) states that if workers can be sequenced from slowest to fastest so that each worker is strictly faster than his predecessor at every point along the production line, then there is a stable fixed point that the system will converge to, independent of the initial positions of the workers. Bartholdi *et al.* (2001) addressed the case of stochastic processing times, and proved a similarity between the deterministic and stochastic systems as the number of stations goes to infinity.

Bartholdi *et al.* (1999) focused on two- and three-worker bucket brigade production lines and described all possible asymptotic behavior as a function of the workers' relative speeds, which is assumed to be constant over the entire line. For two workers, the authors showed that only two modes of asymptotic behavior exists: a fixed point with optimal production rate or a period-two orbit with suboptimal production rate. In this paper, we study systems in which workers have varying levels of specialization at different tasks. That is, a worker's speed at a particular task depends more on the type of the task, rather than the worker's skill profile. Hence, a worker maybe faster on one portion of the line and significantly slower on another.

There are several real life systems in which workers have different speeds in different portions of the line. Systems with stations that require extensive specialization or training frequently have this type of structure for worker speeds. Also, systems with high labor turnover frequently encounter situations where newly hired workers have skills that are applicable only to parts of the production line (Hutchinson *et al.*, 1997).

In this paper, we provide some basic principles on bucket brigades in which worker speeds do not dominate each other uniformly. We analyze the dynamics and performance (throughput) of such systems under two operating rules. The first one of these environments is one in which sufficient machinery exists so that passing or overtaking of the downstream worker by an upstream worker is allowed. Warehouse order picking seems to be a prototypical case for this assumption. To our knowledge, this represents the first analysis of such a bucket brigade environment. Following the passing environment, we provide the analysis of bucket brigade environments in which passing is not allowed, hence a worker gets blocked if he/she catches up with the downstream worker.

We present the analysis of these cases for two worker bucket brigade systems. Our results provide a basis to study systems with more workers and other complicating assumptions. We note, however, that our analysis would be useful to many real life systems since many bucket brigade systems consist of three or two workers (Bartholdi *et al.*, 1999).

The next section outlines our assumptions and provides a formalization of our model of worker speeds. In Section 3 we provide analysis of the bucket brigade with passing and present general results on the behavior. Section 4 includes similar results for the dynamics of bucket brigades under the assumption of blocking. Finally, in Section 5 we provide some concluding remarks and insights, and suggest some areas for future work.

2 Modeling Assumptions

Our basic model and assumptions are similar to those made by Bartholdi and Eisenstein (1996). We consider a production line with fully cross-trained workers and continuous tasks. The processing times are assumed to be deterministic and each worker has a velocity function that gives his/her instantaneous velocity at each point along the production line. Workers walk back to take over jobs with infinite velocity (zero walk-back times). However, our work differs from that of Bartholdi and Eisenstein (1996) in the sense that we consider systems in which the velocity of one worker does not uniformly dominate that of another at every point along the production line.

Specifically, we consider a bucket brigade with two workers, which we denote as worker A and worker B. For convenience, we refer to worker A as a male and worker B as a female. We scale work content and time such that worker B has a uniform speed along the production line which we set to 1. Work content of a job (i.e., the amount of work already performed on the job) is denoted by x where $x = 0$ reflects a new entry into the production line and $x = 1$ corresponds to a finished product. We assume that the speed of worker A is a function of his location on the line. To ensure that worker B's speed does not uniformly dominate that of worker A, we assume that worker A has speed c_1 on the interval $[0, X)$ and c_2 on the interval $[X, 1]$ with $c_1 < 1 < c_2$.

Our assumptions on the worker speeds may appear restrictive at first. However, note that this assumption of piecewise constant worker speeds is made here to make the presentation simpler, and will be generalized to a velocity function, $c(x) > 0$ later on. Our model of worker speeds covers all general instances of worker speeds with a single break point. That is, with appropriate scaling and renaming of the workers, the models adequately represent any system in which worker speeds have different relative values in different regions of work.

We are interested in two major issues: (i) which arrangement or sequencing of the workers will lead to a self organizing production line, (i.e., a line that is stationary in some sense from almost all initial conditions), and (ii) which arrangement leads to the best average throughput. Throughout the paper, we will assume that work is continuous in a similar spirit to Bartholdi and Eisenstein (1996), however, we note that the analysis under the discrete work or discrete workstations case yields similar results, details of which are omitted. For each of the passing and blocking cases, we provide the analysis of the dynamics and compare the performance of the bucket brigade with more traditional worker assignment schemes.

3 Dynamics for the Passing Case

As discussed above, prior work on bucket brigades generally assumes that workers get blocked. In this section, we extend the term bucket brigade for environments in which passing or overtaking a downstream worker is an option. While this may be an unrealistic assumption for traditional assembly lines with a single piece of equipment, there are many environments in which this may be the more appropriate assumption. Such environments include warehouses (Bartholdi and Eisenstein, 1996), systems with general purpose equipment, etc. Throughout this section, we assume that there is sufficient machinery or equipment in the line (or sufficient walking space as in the case of warehouses) so that workers are allowed to pass each other and not get blocked by the downstream worker occupying a station.

Our assumption of varying worker speeds implies that one worker does not dominate the speed of the other on all sections of the work. Hence, an ordering of the workers based on their speeds is not well defined. For this reason, we consider both possible ways of ordering the workers: worker A at the end (i.e., downstream) and worker B at the end.

To provide intuition on our analysis, the following section includes a case study for one of these cases under the passing case. The detailed analysis of the worker B at the end case is omitted for brevity, but similar results for that case can be found in §3.3.

3.1 Case Study: Worker A at the End

First, let's suppose that worker A is placed at the end of the line. Whenever worker A finishes his job, he goes back and takes over a job from worker B, wherever she is. It is assumed that worker B can pass worker A for some time. However, we assume that worker A is fast enough near the end of the production line such that he always finishes his job first. We define $q_B(t)$ ($q_A(t)$) as the position of worker B (worker A) along the production line at time t. Since the speed of worker B speed is equal to 1 for all portions of the work we find

$$q_B(t) = t \text{ for all } t \geq 0 \ .$$

For worker A, $q_A(t)$ will change as a function of where he started. Let q^0 denote the location that worker A took over the first job, that is, $q^0 := q_A(0)$. If worker A starts at a point after the break point X, (i.e., $q^0 > X$), then the location of worker A at time t will be governed by:

$$q_A(t) = q^0 + c_2 t \text{ for all } t \geq 0 \ .$$

If worker A takes over the job at some point before X (i.e., $q^0 \leq X$) then we get:

$$q_A(t) = \begin{cases} q^0 + c_1 t & \text{for } t < t_X, \\ X + c_2(t - t_X) & \text{for } t \geq t_X, \end{cases}$$

where t_X is the time it takes worker A to get to $x = X$:

$$t_X = \frac{X - q^0}{c_1} \ .$$

It takes worker A $\bar{t}_1$ time units to complete the first job, which is found by setting $q_A(t) = 1$.

$$\bar{t}_1 = \begin{cases} \frac{1-q^0}{c_2} & \text{for } q^0 > X, \\ t_X + \frac{1-X}{c_2} = X(\frac{1}{c_1} - \frac{1}{c_2}) - \frac{q^0}{c_1} + \frac{1}{c_2} & \text{for } q^0 \leq X. \end{cases}$$

In general, $\bar{t}_n$ is the amount of time that worker A spends to process job n. After the completion of a job, say job n, the new starting point for worker A is the point where worker B is at time $\bar{t}_n$ assuming that we reset time whenever a job is finished. Hence, we end up with a piecewise linear map of the form $q^{n+1} = f(q^n)$ given by

$$q^{n+1} := q_B(\bar{t}_n) = \begin{cases} \frac{1-q^n}{c_2} & \text{for } q^n > X \\ X(\frac{1}{c_1} - \frac{1}{c_2}) - \frac{q^n}{c_1} + \frac{1}{c_2} & \text{for } q^n \leq X. \end{cases} \quad (1)$$

The map is piecewise linear with slopes $-1/c_1 < -1$ and $-1/c_2 > -1$. The elbow of the map is at $q^n = X$ with $f(X) = (1 - X)/c_2$.

To analyze this map further, we need a few key definitions from dynamical systems theory, and their implications for the behavior of the bucket brigade.

- For any map $x_{n+1} = f(x_n)$ we call $x = x_s$ a fixed point if $f(x_s) = x_s$. Graphically this implies that the graph of $f(x)$ intersects the diagonal $x_{n+1} = x_n$ at x_s. For a two worker bucket brigade a fixed point implies a starting point x_s for the second worker such that the first worker always works from 0 to x_s and the second worker from x_s to 1.

- A point p_1 is called a period two point if $f(f(p)) = p$. The corresponding orbit $p \to f(p) := q \to f^2(p) = f(q) = p \to ...$ is called a period-two orbit. Graphically this implies that the graph of $f^2(x)$ intersects the diagonal at two points p and q different from the fixed point. A periodic orbit in a two worker bucket brigade implies that handover of the product from the first to the second worker happens alternatingly at position p and at position q.

- A fixed point x_s is called (asymptotically) stable if there exists a neighborhood of the fixed point such that all initial conditions inside this neighborhood approach the fixed point under the iterations of the map $f(\cdot)$. It is a theorem in dynamical systems that x_s is stable if the slope $|f'(x_s)| < 1$ (Alligood *et al.*, 1996). A fixed point is globally stable if all initial conditions get attracted to it. A two worker bucket brigade has a globally stable fixed point if for every starting point of the second worker they eventually end up working the same part of the production line. We call such a system self balanced. Similarly, a period-two orbit $p \to q \to p$ is stable if p and q are stable fixed points of $f^2(\cdot)$. We call a bucket brigade that has a stable period two-orbit "self organized".

The most important feature of the map (1) is that there always exists a fixed point x_s where $f(x_s) = x_s$. If worker A starts at x_s, he will take exactly the same amount of time to finish as the time that worker B will take to get to x_s. Hence the handoff will always happen there.

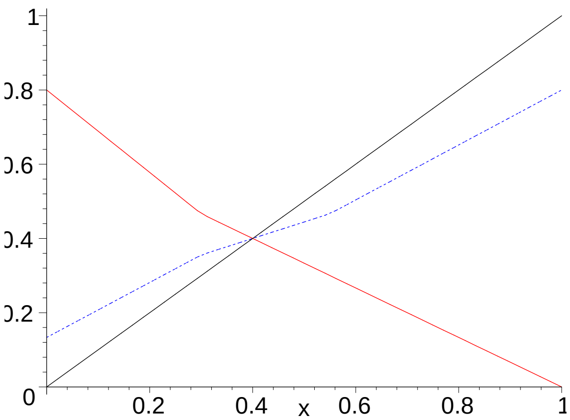
If $X \leq x_s$ (i.e., slope at x_s is equal to $-1/c_2 > -1$), then this critical position represents a stable fixed point, which means that for all or a subset of initial conditions, the bucket brigade self-balances itself and worker B hands off her job to worker A at this critical position at every iteration. From (1), we can calculate this stable fixed point as $x_s = \frac{1}{1+c_2}$. If on the other hand, $X > x_s$, the fixed point $x_s = \frac{X(c_2-c_1)+c_1}{c_2(1+c_1)}$ is unstable.

Figure 1 shows all four possible cases for the dynamics of the bucket brigade. Each of the subfigures shows: (i) the Poincaré map, $q^{n+1} = f(q^n)$ (depicted by the dashed line), (ii) $q^{n+2} = f^2(q^n) = f(f(q^n))$, and (iii) $y = x$ line as the diagonal. Figures 1(a) and 1(b) show the case of $X \leq x_s$, which results in a stable fixed point. Figures 1(c) and 1(d) show the case of $X > x_s$, which results in an unstable fixed point. The critical quantity for the dynamics in both cases is M, which is defined as the time that worker A would need to get through the whole production line. Then, $M = f(0)$ can be found as

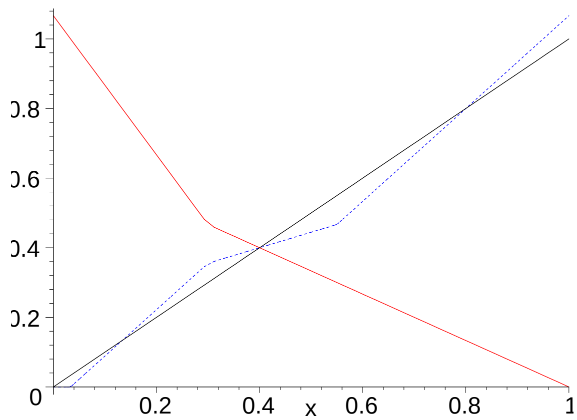
$$M = \frac{X}{c_1} + \frac{1-X}{c_2} \ . \tag{2}$$

Note that if $M > 1$, then worker B is **faster on average** than worker A.

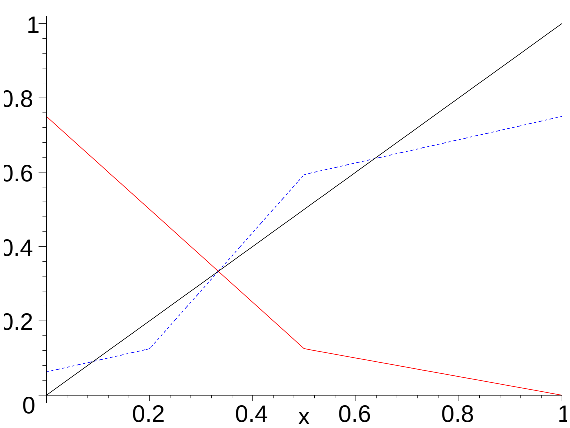
- $X \leq x_s$, stable fixed point:
  - Figure 1(a) shows the case with worker A faster on average than worker B ($M \leq 1$). We see that the stable fixed point attracts for all initial conditions (i.e., independent of where worker A starts from, assuming that worker B always starts at point zero).



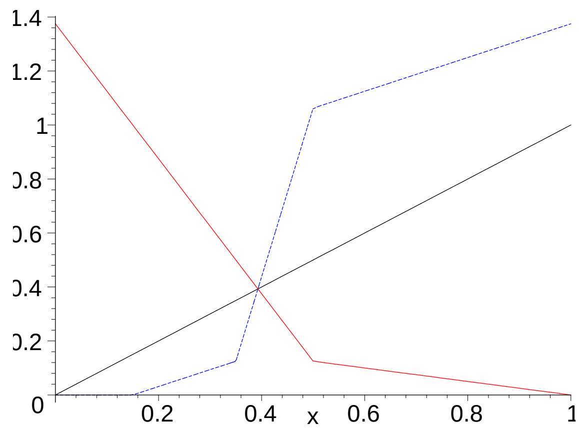
(a) $X \leq x_s,\ M \leq 1$



(b) $X \leq x_s,\ M > 1$



(c) $X > x_s,\ M \leq 1$



(d) $X > x_s,\ M > 1$

Figure 1: All possible dynamics for the passing case when worker A is placed at the end

- Figure 1(b) shows the case with worker B faster on average than worker A ($M > 1$). In this case, the fixed point is still stable, but there also exists a period-two orbit that is unstable. That is, all initial conditions inside the period-two orbit converge to the fixed point and all initial conditions outside the period-two orbit lead to a situation where worker B finishes her job before worker A does, which means that A is not fast enough at the end of the production line to maintain the ordering.

- $X > x_s$, unstable fixed point:
  - Figure 1(c) shows the case with worker A faster than worker B on average ($M \leq 1$). We see that the fixed point is unstable, however there exists a stable period-two orbit that attracts all initial conditions and preserves the order of the bucket brigade.
  - Figure 1(d) shows the case $X > x_s$ with worker B faster than worker A on average ($M > 1$). In this case, the fixed point is unstable and all initial conditions eventually lead to a situation where worker B finishes her job before worker A does and hence, the order of the bucket brigade is not preserved.

3.2 A New Rule that Yields Self-organization

Our current rules assign a fixed starting worker and a fixed finishing worker. However, for some speed configurations the starting worker will finish first and hence, the line becomes unbalanced. We augment our bucket brigade rules and stipulate that in this case, the workers are allowed to reverse the order of the starter and finisher. With this new rule, the bucket brigade becomes self-organizing to either a stable fixed point or a periodic orbit without takeovers ever happening again for all initial conditions.

As an example, consider the following. Assume worker B is positioned at the end of the line, and she is slower than worker A on the average (i.e., $M < 1$). We will prove below that we then have a stable fixed point surrounded by an unstable periodic orbit that generates two distinct intervals of initial positions that do not converge to the fixed point and for which worker A will catch up with worker B. Now, due to this new rule, as soon as worker A catches up with worker B and completes before worker B, worker A goes back and takes worker B's job, which reverses the order of the workers. Hence, worker A is now at the end and the results for that case for $M < 1$ apply. We showed above that for $M < 1$ and worker A at the end, we have a stable periodic orbit for all initial conditions. Hence, the bucket brigade balances to a periodic orbit.

Similarly, for $M > 1$ and worker A at the end, the fixed point is unstable and worker B will catch up with worker A for some initial conditions. Now, due to the new rule, as soon as worker B catches up with worker A, the order of the workers is reversed. We will prove in the next section that the new configuration has a stable fixed point for all initial conditions.

Note that this reversal will convert a bucket brigade with an unstable fixed point into one with a stable fixed point. The fixed point has the optimal throughput for each given worker ordering. However, as we demonstrate with the following example, it is very important to note that the stable fixed point may have a lower throughput than the unstable one.

Our analysis and results have important implications for systems with only partial cross-training. Suppose, for example, that worker A can only process $(X, 1)$. We can model the speed of worker A to be almost zero ($c_1 = \epsilon > 0$), which fits our assumptions on the worker speeds. Now since $M > 1$, worker A is slower on average than worker B. Suppose that worker A, who is only partially cross-trained, is placed at the end of the line. In this case, the dynamics will lead to an unstable fixed point at $x_s < X$, and as stated above, all initial conditions eventually lead to a situation where worker B finishes her job before worker A does. If one implements the new rule and reverses the order of the workers, the bucket brigade balances at a globally stable fixed point, but this fixed point will be very close to 0. In other words, worker A will be doing almost nothing, and worker B will be processing each job from almost start to finish.

3.3 General Results for the Passing Case

The previous case study can be generalized into theorems that apply for both worker orders, A at the end and B at the end, respectively. In the following, we assume that worker A has constant speeds c_1 on the interval $[0, X)$ and c_2 on the interval $[X, 1]$ with $c_1 < 1 < c_2$. Theorem 2 extends the results to the case of a general velocity function with a single breakpoint, $0 < X < 1$. The term slower (or faster) worker refers to the worker that is slower (faster) on average. We typically present proofs for the case of worker A at the end. The proofs for B at the end are similar and can be obtained from the authors.

Lemma 1 *There always exists a unique fixed point that balances the bucket brigade.*

Proof: Assume both workers start at zero. Then, there are two possible cases for finishing a product: (i) the worker that is faster on average is assigned the end position and hence, the original order is maintained, (ii) the slower worker is assigned the end position and hence, the upstream worker finishes his/her job first (which can happen since passing is allowed). Note that even if the slower worker is assigned to the end, the faster worker may not be able to catch up with him/her if the starting position of the slower worker is sufficiently far down the line. This implies that there is a starting position $q > 0$ for the slower worker such that he/she will finish together with the faster worker starting at zero. Hence, the Poincaré map $f(x)$ has $f(0) < 1$ in case (i) and $f(q) = 1$ in case (ii). Since $f(1) = 0$ in all cases, and since f is continuous, by the Intermediate Value Theorem there exists a point x_s such that $f(x_s) = x_s$. Finally, since f is always monotonously decreasing, the fixed point is unique.

Lemma 2 *For worker A (B) at the end, the fixed point is stable if it is located in the high (low) speed region. Otherwise, the fixed point is unstable.*

Proof: For the case of worker A at the end, the slopes of the Poincaré map are given by $-\frac{1}{c_1} < -1$ and $-\frac{1}{c_2} > -1$ (see (1)). Hence, any fixed point in the high speed region of worker A is stable.

For the case of worker B at the end, the slopes of the Poincaré map are $-c_1 > -1$ and $-c_2 < -1$. Hence, any fixed point in the low speed region of worker A is stable. Note also that the eigenvalues are always negative and hence, the map oscillates around the fixed point.

Theorem 1 *The following results hold true for the dynamics of two-worker bucket brigades with passing:*

1. *If worker A (B) is at the end and is on average the faster worker and if the fixed point is stable, then it attracts all initial conditions and the line balances itself.*

2. *If worker A (B) is at the end and is on average the faster worker and if the fixed point is unstable, then there exists a stable period-two orbit that attracts all initial conditions.*

3. *If worker A (B) is at the end and is on average the slower worker and if the fixed point is stable then there exists an unstable period-two orbit. All initial conditions inside the period-two orbit lead to a balanced line at the fixed point, initial conditions outside the period-two orbit lead to a reversal of the finishing order of the bucket brigade.*

4. *If worker A (B) is at the end and is on average the slower worker and if the fixed point is unstable then all initial conditions lead to a reversal of the finishing order of the bucket brigade.*

Proof: Consider the case of worker A at the end. The proof is based on two facts:

(i) If the worker that is fastest on average is at the end then the Poincaré map $f(x)$ is defined for all initial conditions and in particular $0 < f(0) < 1$ while $f(1) = 0$.

(ii) Recall that $x = X$ is the point where the velocity order switches. We find that $f^2(X) > X$ and $f^2(Y) < Y$ where Y the preimage of X. If the fixed point is stable then $Y > X$, if it is unstable we have $X > Y$. The second iteration of the Poincaré map is piecewise linear and consists of three line segments with the slopes $(c_1c_2)^{-1}, c_2^{-2}, (c_1c_2)^{-1}$, respectively. The end points of these line segments are $f^2(0), f^2(X), f^2(Y)$ and $f^2(1)$ (see Figure 1).

Simple geometry and the Intermediate Value Theorem implies all the statements.

Theorem 2 *All previous theorems are still valid if the velocity of worker A is not piecewise constant but depends on its exact location along the production line, as long as there is only* **one** *point at which the relative order of the workers' velocities changes. That is, the velocity of worker A satisfies* $c(x) < 1$ *for* $x < X$ *and* $c(x) > 1$ *for* $x > X$.

Proof: Assume a speed distribution for the worker A of the form

$$c(x) = \begin{cases} c_1(x) & < 1 \quad \text{for} \quad x < X \\ c_2(x) & > 1 \quad \text{for} \quad x > X. \end{cases} \tag{3}$$

The position of worker A is given through the implicit solution of the differential equation $q_A'(t) = c_1(q_A(t))$. For $q_0 < X$ we get

$$\begin{aligned} \int_{q_0}^{x(t)} \frac{d\xi}{c_1(\xi)} &= t \qquad\qquad t < t_X, \\ X + \int_{X}^{x(t)} \frac{d\xi}{c_2(\xi)} &= t \qquad\qquad \text{for } t \geq t_X, \end{aligned}$$

where t_X is the solution of the equation $\int_{q_0}^{X} \frac{d\xi}{c_1(\xi)} = t_X$. The finishing time $\bar{t}$ is similarly given as the solution of the equation $X + \int_X^1 \frac{d\xi}{c_2(\xi)} = \bar{t}$ for $q_0 < X$ and $\int_{q_0}^1 \frac{d\xi}{c_2(\xi)} = \bar{t}$ for $q_0 > X$. The Poincaré map $q^{n+1} = f(q^n)$ is still given as

$$q^{n+1} := q_B(\bar{t}_n) = \bar{t}_n. \tag{4}$$

Note that

- $f(0) = \int_0^X \frac{d\xi}{c_1(\xi)} + \int_X^1 \frac{d\xi}{c_2(\xi)}$ is still the average time for worker A on the production line. Hence, if worker A is faster on average than worker B, then $f(0) < 1$, whereas $f(0) > 1$, if worker B is faster than worker A on average. Since $f(1) = 0$, the arguments for Theorem 1 still hold.

- Uniqueness and stability of the fixed point can easily be determined by noting that

$$\frac{d\bar{t}}{dq^n} = \begin{cases} -\frac{1}{c_1(q^n)} & \text{for } q^n < X \ , \\ -\frac{1}{c_2(q^n)} & \text{for } q^n > X \ . \end{cases} \tag{5}$$

  Hence the Poincaré map is monotonously decreasing and fixed points $x_s < X$ are unstable and those for $x_s > X$ are stable which proves the generalization of Theorem 2.

The geometric arguments of Theorem 1 continue to hold if the Poincaré map is not piecewise linear but is only monotonously decreasing with a slope that is less than -1 for $q^n < X$ and between 0 and -1 for $q^n > X$.

Lemma 3 *The throughput for a period-two orbit is always less than the throughput of the fixed point.*

Proof: Consider again the case of worker A at the end. By construction of the Poincaré map (4) we have that $f(q) = \bar{t}$, the time between successive finishes. Hence, to prove the theorem we need to show that for a periodic orbit $p_1 \to p_2 \to p_1 ...$ it takes longer to finish the periodic orbit than to finish two products when starting at the fixed point, i.e. $p_1 + p_2 > 2x_s$.

Consider Figure 1(c). Since $f(p_1) = p_2$, the interval $[p_1, x_s]$ is mapped to the interval $[x_s, p_2]$ by f. However, f is piecewise linear with a slope less than -1 on that interval and hence $x_s - p_1 < p_2 - x_s$, which proves the claim. A similar argument holds for an unstable periodic orbit as can be seen in Figure 1(b). f is contracting on the interval $[x_s, p_2]$ and hence the statement again holds.

Lemma 4 *A reversal of worker order for given functions $c_1(x)$, $c_2(x)$ and X will lead to a change in stability of the fixed point.*

Proof: The proof involves simple but tedious algebra, details of which can be obtained from the authors.

We note that the last theorem together with Theorem 1 may be used to pursue different strategies: If global self organizing is desired, then one needs to choose the on-average-faster worker at the end. In that case, the bucket brigade will either balance on the fixed point or self organize on the periodic orbit. On the other hand, if optimal throughput is desired then one may have to change the order of the workers or may have to go to a fixed work allocation near the unstable fixed point. The tradeoff is that not all initial conditions will lead to a balanced production line.

Another important insight from our results is related to the implications of passing on the machine or equipment requirements. Notice that passing will not occur at a fixed point, and may or may not occur at a stable periodic orbit (depending on the detailed values of c_1 and c_2). If *global* self-organizing behavior is important then passing cannot be avoided and space or machinery will have to be provided for that situation. However, if the fixed point or the periodic orbit can be located approximately, then only *local* stability may be sufficient. In this case, one does not need to duplicate machinery and invest in a second set of equipment for the whole line.

4 Dynamics for the Blocking Case

In this section we assume that tools or equipment in the line are not duplicated, so that workers cannot pass each other. That is, a worker becomes blocked if he/she catches up with the downstream worker. This is the original assumption used by Bartholdi and Eisenstein (1996) to study apparel manufacturing lines and warehouses. For many systems with specialized equipment this becomes a more appropriate assumption due to increased costs of duplicate tooling.

Under the assumption of continuous tasks, the concept of blocking is different than usual. Rather than idling and waiting for the other worker to leave a workstation, if a faster worker is blocked behind another (slower) worker, then he/she is forced to work at the speed of the leading worker. The basic assumption is that worker A and worker B can get arbitrarily close and still work on their own tasks. Having stated this difference, we note that the analysis of dynamics of systems with discrete tasks yields similar results and is omitted from this manuscript for brevity.

4.1 Case Study: Worker A at the End

Again, we start with a case study that demonstrates the method and consider the case in which worker A is placed at the end of the line. For this case study, we assume that the breakpoint at which the worker A's speed changes, X, is equal to 1/2. The results are generalized in §4.2.

Suppose that worker A starts at $q^0 \geq 1/2$. In that case, since worker A is faster on the second half of the line, worker B will never catch up with worker A and hence, will not be blocked. Then, for all $t > 0$, the dynamics of the bucket brigade for $q^0 \geq 1/2$ is given by:

$$\begin{aligned} q_B(t) &= t \ , \\ q_A(t) &= q^0 + c_2 t \ . \end{aligned}$$

If worker A starts on the first half of the line (i.e., $q^0 < 1/2$), there may or may not be blocking depending on worker A's relative speed, c_1 and the starting point, q^0. For all $t > 0$ such that $q_A(t) > q_B(t)$ the dynamics are:

$$\begin{aligned} q_B(t) &= t \ , \\ q_A(t) &= q^0 + c_1 t. \end{aligned}$$

Worker B gets blocked when $q_A(t) = q_B(t)$, which happens at $t_{block} := q^0/(1-c_1)$. Since $q_A(t) = 1/2$ at $t = t_{1/2}$ we have no blocking if $t_{1/2} < t_{block}$; that is, if $q^0 > (1/2)(1-c_1) =: q_{block}$. Therefore, if $q_{block} < q^0 < 1/2$ then

$$\begin{aligned} q_B(t) &= t \ , \\ q_A(t) &= \begin{cases} q^0 + c_1 t & \text{for } t < t_{1/2} \\ 1/2 + c_2(t - t_{1/2}) & \text{for } t \geq t_{1/2} \ , \end{cases} \end{aligned}$$

and, if $q^0 \leq q_{block}$, then

$$q_B(t) = \begin{cases} t & \text{for } t < t_{block} \\ \frac{q^0}{1-c_1} + c_1(t - t_{block}) & \text{for } t_{block} \leq t < t_{1/2} \\ 1/2 + t - t_{1/2} & \text{for } t \geq t_{1/2} \ , \end{cases}$$

$$q_A(t) = \begin{cases} q^0 + c_1 t & \text{for } t < t_{block} \\ \frac{q^0}{1-c_1} + c_1(t - t_{block}) & \text{for } t_{block} \le t < t_{1/2} \\ 1/2 + c_2(t - t_{1/2}) & \text{for } t \ge t_{1/2}\ . \end{cases}$$

The time that it takes worker A to finish the first job, $\bar{t}_1$ (i.e., $q_A(\bar{t}_1) = 1$) is equal to

$$\bar{t}_1 = \begin{cases} \frac{1}{2c_2} + t_{1/2} & \text{for } q^0 < 1/2 \\ \frac{1-q^0}{c_2} & \text{for } q^0 \ge 1/2. \end{cases}$$

The new starting point for worker A to take over job $n+1$ is the point where worker B is at time $\bar{t}_n$ assuming that we reset time whenever a job is finished. Hence, the piecewise linear map of $q^{n+1} = f(q^n)$ is given by

$$q_{n+1} := q_B(\bar{t}_n) = \begin{cases} \frac{1}{2} + \frac{1}{2c_2} & \text{for } q^n < q_{block} \\ \frac{1}{2}(\frac{1}{c_1} + \frac{1}{c_2}) - \frac{q^n}{c_1} & \text{for } q_{block} \le q^n < \frac{1}{2} \\ \frac{1-q^n}{c_2} & \text{for } \frac{1}{2} \le q^n. \end{cases} \quad (6)$$

Figure 3(b) shows a typical map. Map (6) has a fixed point at

$$x_s = \frac{c_1 + c_2}{2c_2(1 + c_1)}, \quad (7)$$

which is always less than 1/2. The slope of the map at that point is $-1/c_1$ and hence, the fixed point is always unstable. However, there exists a stable period-two orbit oscillating between the two points, p_1 and p_2

$$p_1 = \frac{c_2 - 1}{2c_2^2}, \quad (8)$$

$$p_2 = \frac{c_2 + 1}{2c_2}. \quad (9)$$

Since the periodic orbit is stable for all initial conditions, the line balances itself to the periodic orbit. The throughput of the bucket brigade in this case is given by

$$TP_{block} = \frac{2}{p_1 + p_2} = \frac{4c_2^2}{c_2^2 + 2c_2 - 1}\ . \quad (10)$$

Figure 2 graphs the throughput of the bucket brigade under the blocking and passing assumptions (the lighter color and darker color curves on the top) as well as that of the parallel lines (bottom curve) for $X = 1/2$ and for all $\{(c_1, c_2) | 0.6 \le c_1 \le 0.99, M < 1\}$. Note that when $M < 1$ and worker A is at the end, the passing assumption results in a stable periodic orbit, which makes our comparison a fair one. The throughput of the stable periodic orbit for $X = 1/2$ and $M < 1$ can be found as

$$TP_{p_1,p_2} = \frac{2}{p_1 + p_2} = \frac{4c_2(c_1c_2 - 1)}{3c_1c_2 - 3c_2 - c_1 + c_2^2}\ . \quad (11)$$

Figure 2 shows that the throughput in the blocking case is very similar to that under the passing assumption (and significantly higher than that of the parallel case) for all values of c_1 and

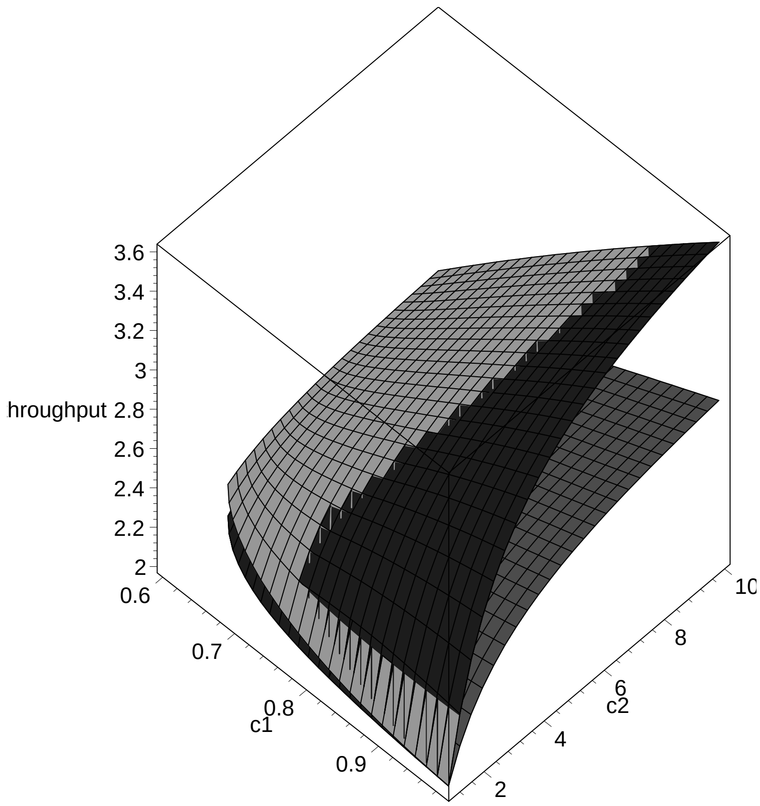


Figure 2: Throughput on a grid of c_1 and c_2 for parallel workers, bucket brigade with passing in the oscillatory case and for the oscillatory case with blocking

c_2. However, for small values of c_1, the blocking case has higher throughput than the passing case, whereas for c_1 close to 1, the passing case has better throughput. The boundary between the two cases is characterized by the fact that blocking occurs for the periodic orbit. For small c_1, blocking happens at the point p_1 while for $c_1 \to 1$ no blocking occurs. This leads to the counterintuitive situation that a system that slows down a worker for a certain time (i.e., due to blocking) has a higher throughput than a system that allows the same workers to always work at their speed and pass each other. The reason for this can be understood by inspection of the behavior of the periodic points as c_1 decreases. In the blocking case, the periodic points do not depend on c_1 while in the passing case we have that for decreasing c_1, $p_1 \to 0$ and $p_2 \to 1$. This forces the worker A to work on a large part of the production line where he is very slow. Note that this result also implies that the investment of a dedicated production line for each worker under this type of speed structure will not necessarily improve the overall throughput.

4.2 General Results for the Blocking Case

Under the assumption of blocking, the dynamics for the cases of worker A at the end and worker B at the end are drastically different. Hence, we will present the results of those two cases separately. Assume that worker A is at the end. Since worker A is faster for $x > X$, he will always finish first. Also, whenever blocking occurs the next handoff is always at $x = X + \frac{1-X}{c_2}$, no matter how long worker B was blocked before. We can derive the following theorems.

Lemma 5 *Blocking occurs whenever* $q_0 < X(1 - c_1)$.

Proof: Blocking occurs if worker B arrives at X earlier than worker A. We solve $t_X > X$ for q_0 to get the result.

Corollary 1 *The fixed point never gets blocked.*

Proof: By definition, the fixed point is the point where worker B is when worker A finishes, which implies the corollary.

Theorem 3 *There exist three qualitatively different regimes for the blocking case with worker A at the end:*

1. *A globally stable fixed point.*
2. *A unstable fixed point surrounded by a globally stable periodic orbit. The periodic orbit may or may not involve blocking at every second iteration.*
3. *A stable fixed point surrounded by an unstable periodic orbit surrounded by a stable periodic orbit. The stable periodic orbit involves blocking at every second iteration.*

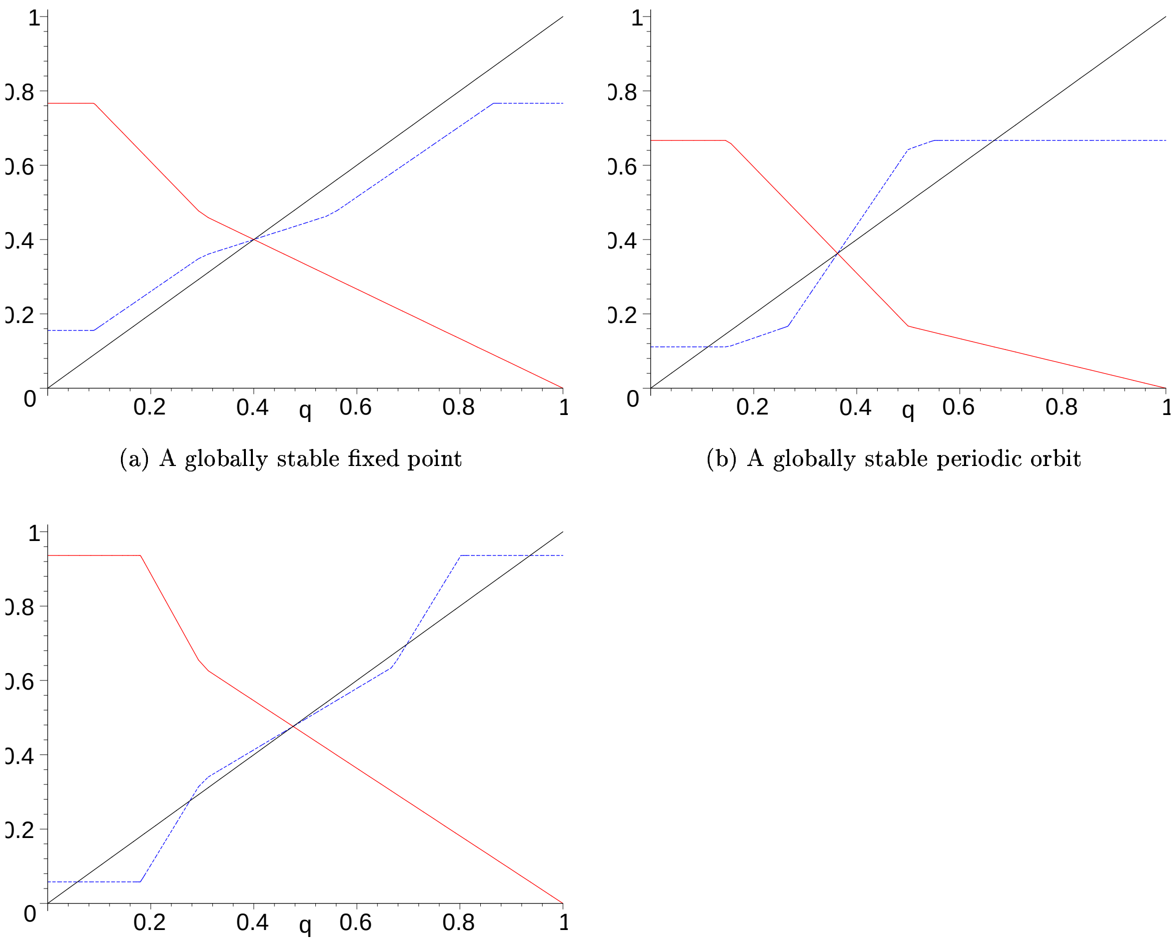
Proof: Figure 3(a) through 3(c) represent the three cases given in (i), (ii) and (iii), respectively. The argument is very similar to that given in the proof of Theorem 1, and is omitted for brevity.

Let's now assume that worker B is at the end. If B is the faster worker, then no blocking will occur and the results of Theorem 1 apply. If B is the slower worker, then blocking occurs and both workers arrive at the end of the production line together - they act as if they are working on parallel production lines.

Theorem 4 *If worker B is at the end, and she is the slower worker, then there exist two qualitatively different regimes:*

1. *A stable fixed point is surrounded by an unstable periodic orbit, which is surrounded by the stable periodic orbit* $0 \to 1 \to 0 \to 1$.
2. *The unstable fixed point is surrounded by the stable periodic orbit* $0 \to 1 \to 0 \to 1$.

Notice that in the blocking case, the behavior of the production line is **not** strongly determined by the average throughputs of the two workers. That is, the cases of $M > 1$ and $M < 1$ no longer fully characterizes the dynamics of the bucket brigade. However, it is still true that for a given set of parameters, a change of worker order changes the stability of the fixed point. Hence, we can always choose an optimally stable bucket brigade. However, we may have to start sufficiently close to the fixed point to get the bucket brigade to self balance.



(a) A globally stable fixed point

(b) A globally stable periodic orbit

(c) Bistability: both the fixed point and the large periodic orbit are stable

Figure 3: All possible dynamics for the blocking case

5 Conclusions and Further work

We have discussed the most fundamental cases of a bucket brigade with workers having different skills at different parts of a production line, leading to different worker speeds along the line. We find that, if we implement a simple rule (i.e., switch the order of the workers if one worker passes another) the bucket brigade always self organizes itself. We have shown however, that it may not always balance itself on fixed point but may also self organize to a stable period-two orbit. That is, workers hand over jobs at exactly two fixed locations that they visit periodically.

Furthermore, we show that the most complicated dynamics that may happen is a period-two orbit. We found that the crucial parameter is the relative throughput $1/M$ of the two workers and that the overall throughput of the line is optimal for the fixed point, even when that point is unstable.

There are some obvious extensions that we will address in future work:

- What happens if worker A has more than two different speeds, corresponding to his different skills at more than 2 machines? We have not analyzed throughput and the issue of balance of the line in detail yet. However, clearly all maps relating the next starting point to the previous one are always monotonically non-increasing. Hence there is always a fixed point which may or may not be unstable. If the fixed point is unstable, then there exists always at least one stable periodic orbit. Hence nothing more complicated dynamically than a period-two orbit can happen.

- The combinations of relative speeds increases combinatorically if the number of workers increases. Hence, a complete analysis is not feasible. However, we are interested in finding general organizing principles for bucket brigades with task dependent worker speeds. For example,

  - What is the most complicated dynamics for a line with n workers and m different speed levels? In particular, is chaotic dynamics possible?
  - Is an ordering analogous to the two-worker line according to average throughput stabilizing?
  - What happens to a regular (i.e., constant speed along the production line) bucket brigade that is ordered from slowest to fastest when there are two neighboring workers in the middle of the line that are like our two workers, A and B? Under what circumstances they do or don't "disturb" the entire bucket brigade?
  - Assume that there is a a specialized station in the middle of the line, at which only one worker was sufficiently specialized to perform the operations in a fast enough pace. In such a case, should one employ a single bucket brigade, two separate bucket brigades with a stationed worker in the middle or drop the bucket brigade policy altogether?

- Our analysis provides a way to model and understand bucket brigade systems with high labor turnover. In particular, where in the bucket brigade should we place a worker that is just at the beginning of the learning curve? What happens as workers learn and their speeds change dynamically over time?

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