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Dynamics of polar reversals in spherical dynamos

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Structurally stable heteroclinic cycles (SSHCs) are proposed as the mathematical structures that are responsible for the reversals of dipolar magnetic fields in spherical dynamos. The existence of SSHCs involving dipolar magnetic fields generated by convection in a spherical shell for a non-rotating sphere is rigorously proven. The possibility of SSHCs in a rotating shell is proposed, and their existence in a low-dimensional model of the magnetohydrodynamic equations is numerically confirmed. The resulting magnetic time-series shows a dipolar magnetic field, aligned with the rotation axis that intermittently becomes unstable, changes the polar axis, starts rotating, disappears completely and eventually re-establishes itself in its original or opposite direction, chosen randomly.

Keywords: spherical dynamos; reversals; heteroclinic cycles

1. Introduction

Palaeomagnetic studies have shown that over the Earth's history the geomagnetic dipole has not always been oriented as we observe it now. It has often undergone severe inclinations with respect to its present orientation, sometimes followed by complete reversals. These reversals do not occur at fixed intervals of time. They instead show a fundamentally aperiodic behaviour, which has led some scientists in the past to suggest an external cause (like the hit of a meteorite) to explain it. Most geophysicists now admit an internal cause, namely a 'chaotic' behaviour in the physical mechanism that is responsible for the generation and dynamics of the Earth's magnetic field. This problem has therefore become a challenging mathematical question: in the model equations for the geodynamo, does there exist an observable dynamical behaviour compatible with the geodynamo, and, if so, what is its origin? Recent years have seen dramatic progress in dynamo theory. Fully three-dimensional simulations are readily accessible, starting with the pioneering work of Glatzmaier & Roberts (1995), while the first laboratory-sized dynamos are just coming into operation (for a review, see Chossat *et al.* 2001). With the advent of large-scale simulations, there is finally an approach that allows us to go beyond existence proofs of dynamos to the time-evolution of dynamo action. Similarly, once laboratory-sized experiments are routinely run, we can expect a shift of the research focus from existence of dynamo action to the dynamics of the self-generated magnetic field. Specifically, the geomagnetic reversals observed in Glatzmaier & Roberts (1995) and the convection-driven reversal model of Sarson & Jones (1999) deal with the palaeomagnetically established intermittent reversal of the Earth's magnetic field. While these simulations allow us to make interesting physical observations of fluid flow and magnetic fields prior to, during and after the reversal, the small number of events currently observed and the complexity of the dynamics make it hard to make definite statements or proofs about the solution structure of the magnetohydrodynamic equations that lead to or describe the reversal events.

A mathematical approach to model aperiodic reversals of magnetic fields is generated through the concept of structurally stable heteroclinic cycles (SSHCs) (Krupa 1996). These are invariant sets in phase space that have many features that are appealing for a model of geomagnetic reversals.

(i) *Asymptotical stability*. SSHCs can be asymptotically stable or unstable.

(ii) *Structural stability*. There is a large class of perturbations that typically do not destroy the existence of an SSHC. This and asymptotic stability allow for experimental observation of SSHCs.

(iii) *Symmetries* play a crucial role in their existence. We showed in Armbruster & Chossat (1991) the existence of SSHCs due to $O(3)$ symmetry, while Chossat *et al.* (1999*a*) showed that $C_{\infty\mathrm{h}}$ is enough to generate certain types of SSHCs.

(iv) *Intermittency*. Consider an asymptotically stable SSHC that consists of simple fixed points of saddle type. A typical trajectory in the neighbourhood of the SSHC will approach a saddle, stay there for some time and then move along the unstable manifold of the saddle to the next saddle. Since the SSHC is asymptotically stable, after completing one cycle, the orbit is closer to the starting saddle point than before and hence stays longer at that point before moving on to the next one. Ultimately, the inherent noise of the system will determine when the trajectory leaves the fixed points in the cycle. Hence the system is intermittent, with long quasi-stable states interrupted by fast and short-lived excursions between them. Note that while noise clearly is responsible for driving the dynamics away from the fixed points, the size of the noise is completely irrelevant for the existence of intermittency. Since a stable manifold is a mathematical construct, the physical coarse graining of the phase space will not allow a trajectory to stay exactly on the stable manifold, and hence we will always see the heteroclinic dynamics.

Following the discovery that SSHCs can be found in a convection model of a non-rotating Earth (Friedrich & Haken 1986), we suggested in Armbruster & Chossat (1991) that they might be the appropriate mathematical structure to explain the intermittent dynamics of the geodynamo. In the meantime, there has been incremental progress to substantiate this claim. Oprea *et al.* (1997) analyse a numerical simulation of the kinematic dynamo. They drive the induction equation by a time-series representing a heteroclinic cycle generated from a centre manifold analysis of the pure convection problem. They find magnetic reversals. A transversal bifurcation of an SSHC in a rotating planar convection problem to a magnetic SSHC has been studied in Chossat *et al.* (1999*b*). Melbourne *et al.* (2001) have shown the existence of SSHCs in a codimension-3 problem of magnetohydrodynamics for a fast rotating shell with weak symmetry breaking.

The current paper draws several of these approaches together. We rigorously prove the possibility of structurally stable heteroclinic cycles generated by convection in a spherical shell involving the intermittent reversal of dipolar magnetic fields. The heteroclinic cycles are qualitatively different for a non-rotating shell and a rotating shell, respectively. We support our assertions with phase space simulations involving the coupling of eight convection modes and three magnetic modes that show the predicted heteroclinic behaviour. This extends the work of Chossat *et al.* (1999*b*) considerably. In particular, the extension from the planar case of Chossat *et al.* (1999*b*) to the fully spherical case is non-trivial. In addition, we connect the existence of magnetized SSHCs to a known magnetic instability of a convective solution of the spherical Benard problem, whereas Chossat *et al.* (1999*b*) based its analysis on an assumed magnetic instability of two-dimensional convective rolls.

We would nevertheless like to stress that a connection between the polar reversals and the dynamics of the geodynamo is still weak. We can *prove* the existence of SSHCs only in the perturbative limit of the non-rotating spherical dynamo, and we have strong numerical evidence that these SSHCs exist for order-1 rotation rates; however, we cannot, at the moment, extend this analysis to the fast rotator that is the Earth. Hence our analysis may very well apply better to the emerging laboratory dynamos, or especially to the full numerical simulations (Glatzmaier & Roberts 1995), which typically only include relatively slow rotation.

This paper is organized as follows. Section 2 sets up the problem as a convection-driven dynamo problem. Section 3 discusses heteroclinic cycles for Bénard convection in a spherical shell. Sections 4 and 7 contain the main theorems proving the possibility of heteroclinic cycles connecting magnetic states in the non-rotating and rotating case, respectively. Section 5 introduces the low-dimensional model that is used in § 6 to numerically simulate heteroclinic cycles. We conclude in § 8 with a discussion of the relationship between our simulations and the three-dimensional simulations of Sarson & Jones (1999).

2. Set-up of the problem

The mechanism responsible for the dynamo instability can be schematically described as follows. An electrically conducting fluid is flowing in a vessel. Suppose, at an initial time, a magnetic field is present in the vessel. The fluid motion generates an electric current (Ampère's law). Under certain conditions, this electric current will itself induce a magnetic field (Faraday's law) directed in the same direction as the initial one, hence reinforcing it. In order for this mechanism to work, a certain number of conditions must be satisfied. In particular, the electrical energy provided by the magnetic induction must be 'stronger' than the ohmic dissipation in the fluid. Another important condition is that the fluid motion must not violate the so-called 'anti-dynamo theorems' (Moffat 1988). In the classical pre-relativistic setting, the dynamo mechanism is described by the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \frac{1}{\mu\sigma}\Delta \boldsymbol{B} + \nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{B}), \tag{2.1}$$

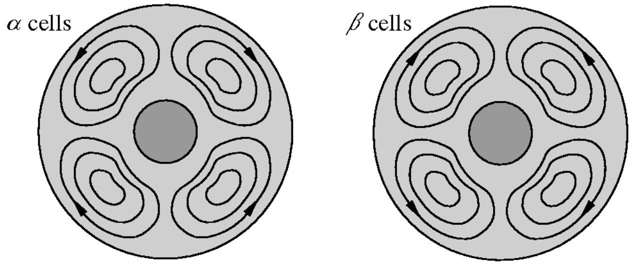

Figure 1. Axisymmetric patterns for $\ell = 2$ modes. The symmetry axis is vertical.

where μ is the magnetic permeability of the fluid and σ its electric resistivity. The number $\eta = 1/\mu\sigma$ is the magnetic diffusivity. The magnetic field is also a divergence-free vector field. The boundary conditions for $\boldsymbol{B}$ can be of various types, depending on the properties of the medium outside the vessel (see Moffat 1988). The relevant non-dimensional number is the magnetic Reynolds number $Re_{\mathrm{m}} = VL/\eta$, where V is a characteristic velocity and L a characteristic length-scale. Clearly, if $\boldsymbol{v} = 0$, corresponding to $Re_{\mathrm{m}} = 0$, no magnetic field can be sustained. Hence one of the basic problems of dynamo theory is to find flows for which a dynamo can exist at finite Re_{m}.

There is obviously a saturation mechanism for this dynamo instability. It is provided by the retroaction of the magnetic field on the fluid motion, which manifests itself through the Lorentz force in the momentum equation. Then the equation of motion of the fluid takes the form

$$\frac{\partial \boldsymbol{v}}{\partial t} = \nu \Delta \boldsymbol{v} + \frac{1}{\rho}\nabla P - (\boldsymbol{v}\nabla)\boldsymbol{v} + \boldsymbol{F} - \frac{1}{\rho}\boldsymbol{B} \wedge \boldsymbol{J}, \tag{2.2}$$

where ν is the viscosity, ρ the density of the fluid and $\boldsymbol{F}$ the external force responsible for the fluid motion. The last term is the Lorentz force. In a convection problem, $\boldsymbol{F}$ is the buoyancy force. In the classical Boussinesq approximation, the fluid is divergence free (incompressible), $\nabla \cdot \boldsymbol{v} = 0$. When the domain is rotating, the Coriolis force $\Omega/\rho \boldsymbol{v} \wedge \boldsymbol{k}$ must be added to $\boldsymbol{F}$ (Ω is the angular rate of rotation and $\boldsymbol{k}$ is the unit vector directed along the axis of rotation). The centrifugal force can be incorporated in the pressure gradient.

In the case we consider here, the fluid vessel is a self-gravitating spherical shell bounded by two rigid spheres of radii $R_{\mathrm{i}} < R_{\mathrm{o}}$. We set $\eta_r = R_{\mathrm{i}}/R_{\mathrm{o}}$. In the absence of rotation, the problem has full spherical symmetry: the equations are invariant under the natural action of the group $O(3)$ induced on the velocity, magnetic, pressure and temperature fields. Rotation reduces the symmetries to the group $C_{\infty \mathrm{h}}$, which is spanned by rotations around the vertical Oz-axis (we choose this axis of rotation) and by the antipodal symmetry σ. Note that we could replace σ by κ_z, the reflection through the horizontal plane $z = 0$. In the following, we shall also denote the reflection through the Oyz-plane and the Oxz-plane, by κ_x and κ_y, respectively. We also introduce the notations ρ_x, ρ_y and ρ_z for the rotations by $\frac{1}{2}\pi$ around the axes Ox, Oy and Oz, respectively. Notice that, for example, $\kappa_z = \rho_x \kappa_y \rho_x^{-1}$.

Another important symmetry in this problem is the 'gauge' symmetry

$$T : \boldsymbol{B} \mapsto -\boldsymbol{B}.$$

We will consider rotation as a perturbation of the purely convective problem. We shall therefore suppose first that $\Omega = 0$. A linear stability analysis shows that the onset of convection ($\boldsymbol{B} = \boldsymbol{0}$) is associated with spherical harmonics $Y_\ell^m(\theta, \phi)$, $-\ell \leqslant m \leqslant \ell$, where ℓ is determined by the aspect ratio η_r. For example, in the case with rigid boundary conditions and $\eta_r = 0.3$, one has $\ell = 2$. Then a classical bifurcation analysis shows that the bifurcated steady states are axisymmetric. Moreover, as a consequence of the Boussinesq approximation, the bifurcation is one sided (supercritical) and, given an axis in $\mathbb{R}^3$, there simultaneously exist two steady states with that axis of symmetry, corresponding to fluid moving in two opposite directions. We call α and β these two steady states with a vertical axis, and, respectively, $\boldsymbol{v}_-$ and $\boldsymbol{v}_+$ the corresponding fluid velocity fields. Figure 1 shows a sketch of the stream lines of these two states. When small departures from self-adjointness are considered, non-axisymmetric states may also bifurcate; however, this is a 'codimension-2' phenomenon, which does not alter the subsequent analysis (see Golubitsky *et al.* 1988).

The first question that we may now ask is the following: can these steady states induce a dynamo action? An answer to this question was given by Vivancos *et al.* (1998), who numerically analysed the induction equation (2.1) when $\boldsymbol{v}$ is the first-order velocity of the bifurcated steady states that occur in spherical Rayleigh–Bénard convection for various aspect ratios η_r. As shown in Vivancos *et al.* (1998), this first-order approximation of the convective state is justified if the magnetic diffusivity is small enough. It can be expected that the dynamo instability still occurs for much higher values of the magnetic diffusivity.

The case with $\eta_r = 0.3$, corresponding to a convective flow dominated by spherical harmonics of order $\ell = 2$, has been carefully investigated in Vivancos *et al.* (1998). Boundary conditions were of rigid type for the fluid and of a perfect conductor for the magnetic field. It was shown that the following hold.

(i) A steady-state dynamo bifurcation occurs for both the $\boldsymbol{v}_-$ and $\boldsymbol{v}_+$ flows, at values 13.25 and 16 of the magnetic Reynolds number, respectively.

(ii) The unstable magnetic modes have the form

$$\boldsymbol{B}(r, \theta, \phi) = (B_r(r, \theta)\cos(\phi - \phi_0), B_\theta(r, \theta)\cos(\phi - \phi_0), B_\phi(r, \theta)\sin(\phi - \phi_0)),$$

where ϕ_0 is an arbitrary phase and has the following symmetry property: it is fixed by the reflection $\kappa_z : \theta \mapsto \pi - \theta$ through the equatorial plane. In other words, one has

$$\begin{aligned} B_r(r, \pi - \theta) &= B_r(r, \theta), \\ B_\theta(r, \pi - \theta) &= -B_\theta(r, \theta), \\ B_\phi(r, \pi - \theta) &= B_\phi(r, \theta). \end{aligned}$$

Since the mode with $\phi_0 = 0$ is clearly invariant under the reflection $\kappa_y : \phi \mapsto -\phi$ through the Oxz-plane, it follows that each mode has a symmetry group spanned by the reflection κ_z and by the reflection through some vertical plane. Note that the antipodal symmetry

$$\sigma : (\theta, \phi) \mapsto (\pi - \theta, \pi + \phi)$$

transforms $\boldsymbol{B}$ to its opposite. Therefore, the complete symmetry group of these modes includes the transformation σT.

The basic states α and β being axisymmetric, this bifurcation is similar to a standard steady-state bifurcation with $O(2)$ symmetry, with one notable exception: the basic states are not isolated, but rather form two-parameter families of solutions, where the parameters are the angles that fix the position of the axis of symmetry. This is an effect of the rotational invariance of the system, and it is well known that neutral modes always exist in this case, which can induce the occurrence of a slow rotational drift of the bifurcated states (see Chossat & Lauterbach 2000). However, as shown in Vivancos *et al.* (1998), the symmetry properties of the linear modes preclude the existence of such a drift. We may therefore conclude that the dynamo bifurcation from the states α and β leads to a circle of new non-axisymmetric steady states (parametrized by ϕ_0). Moreover, as the basic flows $\boldsymbol{v}_{\pm}$ are rotated, the bifurcated solutions form a three-dimensional manifold, parametrized by the angles of the axis of symmetry of the basic convective states and by the phase ϕ_0. Finally, the exchange of stability holds: if the bifurcation is supercritical and if the considered basic state is initially stable, then it loses stability via the dynamo bifurcation, while the bifurcated solutions are stable (in the 'orbital' sense, i.e. the orbit of steady state is an attractor, not each individual point on it).

3. Heteroclinic cycles for Bénard convection in a spherical shell

In this section we review some results that have been obtained successively by Friedrich & Haken (1986), Armbruster & Chossat (1991) and Chossat *et al.* (1999*a*). In order to do this, we first introduce a few basic concepts of bifurcation theory with symmetry. We refer the reader to Chossat & Lauterbach (2000) for a more substantial exposition of this theory. Let

$$\frac{\mathrm{d}v}{\mathrm{d}t} = F(v) \tag{3.1}$$

be a differential equation defined in a (not necessarily finite-dimensional) vector space V, and suppose that a linear action of a symmetry group G (for us, $G = O(3)$) is given in V, such that F is *G-equivariant*,

$$F(g \cdot v) = g \cdot F(v) \quad \forall g \in G,\ v \in V.$$

Now let H be a subgroup of G and set

$$V^H = \{v \in V \mid hv = v\ \forall h \in H\}.$$

Then $F(V^H) \subset V^H$. It follows that if $v(t)$, $t \in I$, is a solution of (3.1) in some open time-interval I, and if, for $t_0 \in I$, $v(t_0)$ has isotropy group H (the isotropy group is the largest subgroup of G that fixes $v(t_0)$), then $v(t)$ lies in V^H for all $t \in I$. This strong constraint imposed by the symmetries of the problem can have important consequences on the dynamics, as we shall now see. We denote by $\mathrm{Fix}(g)$ the space of vectors that are fixed by a given element $g \in G$.

Friedrich & Haken (1986) computed the bifurcation equations (up to order three) for the spherical convection when the aspect ratio η_r is close to a value at which spherical modes with $\ell = 1$ and 2 become simultaneously critical. This value of η_r may depend on the various physical assumptions made in the problem (boundary conditions etc). It is, however, compatible with geophysical data for the convection in the Earth's core. When the parameters Ra (Rayleigh number) and η_r were set in a certain open region close to the critical point in the parameter plane, a numerical integration of these equations showed in most cases the following behaviour.

The system initially places itself into one of the pure $\ell = 2$ steady states, α say. After a while, an $\ell = 1$ axisymmetric disturbance grows, until the system relaxes to the other steady state β. Then the system evolves to an α state, but with an axis of symmetry aligned with either the Ox-or the Oy-axis. This move only involves non-axisymmetric $\ell = 2$ modes. This new state, which we may call α', is a symmetric copy of α and a similar sequence of transitions starts again from α', now involving a state β', which is a copy of β with the axis of symmetry aligned along either Ox or Oy. This behaviour repeats itself indefinitely and aperiodically. Armbruster & Chossat (1991) explained this dynamics by the presence of a *robust heteroclinic cycle* in phase space, which we now describe. The bifurcation equations (or amplitude equations) are defined in a space $V_{1,2}$, which is isomorphic to the direct sum of the irreducible representations of the group $SO(3)$ of degrees $\ell = 1$ and 2. Any vector in that space can be written as

$$v = x_0 Y_1^0 + x_1 Y_1^1 + \bar{x}_1 \bar{Y}_1^1 + y_0 Y_2^0 + y_1 Y_2^1 + \bar{y}_1 \bar{Y}_2^1 + y_2 Y_2^2 + \bar{y}_2 \bar{Y}_2^2,$$

where the Y_ℓ^m are the usual spherical harmonics. The relevant parameters are the Rayleigh number *Ra* and the aspect ratio η_r. By a change of parameters, we can represent (Ra, η_r) through (μ_1, μ_2) such that, when $\mu_1 = 0$, the mode $\ell = 1$ is critical and when $\mu_2 = 0$, the mode $\ell = 2$ is critical. The bifurcation equations take the form of a system of eight differential equations

$$\left.\begin{aligned} \dot{x}_j &= f_j(\mu_1, x_0, x_1, \bar{x}_1, y_0, y_1, \bar{y}_1, y_2, \bar{y}_2), \\ \dot{y}_k &= g_k(\mu_2, x_0, x_1, \bar{x}_1, y_0, y_1, \bar{y}_1, y_2, \bar{y}_2). \end{aligned}\right\} \tag{3.2}$$

Moreover, this system is invariant under the action of $O(3)$ on $V_{1,2}$. In particular, the axisymmetric 'pure' $\ell = 2$ modes y_0 have isotropy group $D_{\infty\mathrm{h}}$, which is the full symmetry group of a cylinder with vertical axis of symmetry. To be more precise, it is spanned by rotations around the Oz-axis, by the rotation by π around an horizontal axis and by the antipodal symmetry σ. Note that we could replace σ by the reflection κ_z.

The y_0-axis is therefore an axis of symmetry in $V_{1,2}$ and is therefore invariant for the differential system defined above. A simple calculation shows that a branch of steady states bifurcates along that axis. Moreover, for our model of convection, the quadratic part of the equations along y_0 has a coefficient close to 0, which implies that the bifurcation is 'almost' supercritical, with two branches (this is a general fact for this class of systems (see Chossat 2001)). The solution with negative y_0 is our state α and the one with positive y_0 is β. Now, it turns out that this axis is itself included in invariant planes P_0 and P_2, which we now define. We set, for convenience, $x_j = x_{j\mathrm{r}} + \mathrm{i} x_{j\mathrm{i}}$ and $y_k = y_{k\mathrm{r}} + \mathrm{i} y_{k\mathrm{i}}$.

(i) $P_0 = \{x_0, y_0\} = V_{1,2}^{C_{\infty\nu}}$, where $C_{\infty\nu}$ is the group spanned by rotations around Oz and by the reflection through a vertical plane.

(ii) $P_2 = \{y_0, y_{2\mathrm{r}}\} = V_{1,2}^{D_{2\mathrm{h}}}$, where $D_{2\mathrm{h}}$ is the group spanned by the rotations by π around Oz and around Ox and by σ.

Both planes contain the y_0-axis. Moreover, P_2 also contains two copies of this axis: $y_{2\mathrm{r}} + \bar{y}_{2\mathrm{r}} = \pm\sqrt{3} y_0$, which correspond to axisymmetric $\ell = 2$ modes with axes aligned

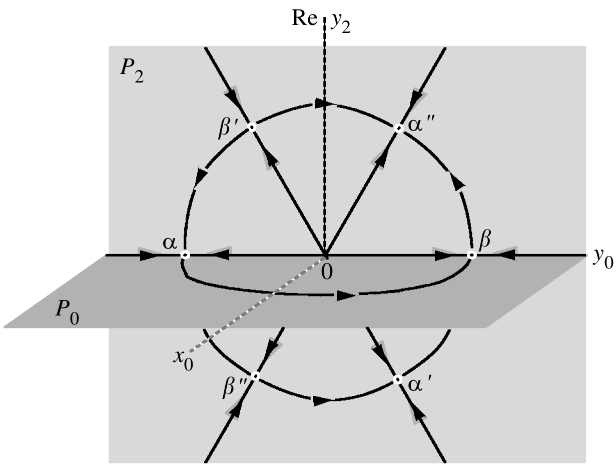

Figure 2. Phase diagram in the planes P_0 and P_2 showing the robust heteroclinic connections between α and β types of steady states.

with Ox and with Oy. In other words, $\alpha' = \rho_y\alpha$, $\alpha'' = \rho_x\alpha$, and similar expressions for the βs. They are also included in higher-dimensional invariant subspaces, which will be described later on.

It was shown in Armbruster & Chossat (1991) that the dynamics in these two planes had the behaviour sketched in figure 2. The heteroclinic orbits are saddle-sink connections within the planes and are therefore robust under $O(3)$-equivariant perturbations. The sequence of connections from α to α' realizes a robust heteroclinic cycle because α' is the same state as α after a rotation of its symmetry axis, and therefore a similar sequence of connections starts from α', which eventually closes back to α.

Other types of robust heteroclinic cycles have also been found by Armbruster & Chossat (1991), notably one connecting steady states to periodic orbits. Chaotic flows have also been found. We shall not consider those flows in the rest of this paper (see, however, Oprea *et al.* (1997) for an approach of the dynamo instability in the presence of such complicated dynamics).

It turns out that the heteroclinic cycle we just described is rarely stable close to the bifurcation, and in fact numerical simulations of Friedrich & Haken have shown that a more complicated attractor, embedding the heteroclinic cycle, was present. This attractor has been elucidated by Chossat *et al.* (1999*a*), who showed the following facts.

(i) Let $S = \mathrm{Fix}(\kappa_z) = \{(x_{1\mathrm{r}}, x_{1\mathrm{i}}, y_0, y_{2\mathrm{r}}, y_{2\mathrm{i}})\}$. There exists an open set of functions f_j, g_k such that the unstable manifold of β is four dimensional, lies in S and is included in the stable manifold of the intersection of the group orbit of α with S (which is a circle R^α). On the other hand, the $\alpha \to \beta$ connection in P_0 is isolated.

(ii) The 'generalized' robust heteroclinic cycle realized by these connections is an attractor for the coefficient values computed by Friedrich & Haken.

Suppose now that one incorporates rotation around a vertical axis into the system. A local bifurcation analysis can still be performed, and the effect of the rotation is simply to add certain terms (corresponding to the Coriolis force) to the equations (3.2). As we have seen in the previous section, these terms break the $O(3)$ symmetry, but are still equivariant by its subgroup $C_{\infty\mathrm{h}}$. At first order, these terms have the form

$$(j\mathrm{i}\epsilon + \gamma_j \epsilon^2)x_j$$

in f_j $(j = 0, 1)$ and

$$(k\mathrm{i}\epsilon + \gamma'_k \epsilon^2)y_k$$

in g_k $(k = 0, 1, 2)$. The coefficients γ_j and γ'_k are negative (see Chossat *et al.* 1999*a*), and ϵ is a parameter proportional to the rate of rotation. We cannot expect that the heteroclinic cycle (even the generalized one) will persist. If, however, the rate of rotation is small enough, a new heteroclinic cycle appears, which connects the α and β steady states together with a rotating wave RW^{α}, which corresponds to a drift induced by the rotation along the circle R^{α} in S (note that S still persists as an invariant subspace). Numerical simulations show that this new object is still an attractor, even for values of ϵ that are not small compared to the values of the bifurcation parameters μ_1 and μ_2. We will come back to this object in § 7.

4. Bifurcation to homoclinic cycles through dynamo instability

In this section we assemble the information provided in the previous sections in order to analyse the combined effect of a dynamo bifurcation from the steady states with $\ell = 2$ and of the presence of an attracting heteroclinic cycle connecting these states. The first remark is that the dynamo bifurcation is a local phenomenon to the purely convective steady states, whereas the heteroclinic connections are global. This implies that this bifurcation does not reduce to a local centre manifold. Nevertheless, as we shall see, the symmetries allow for a rigorous analysis, which results in the following theorem. We assume for the time being that there is no rotation, i.e. $\Omega = 0$ (and hence $\epsilon = 0$). We also assume that the parameters are set so that a robust heteroclinic cycle connecting the α and β types of states exists and is an attractor.

Proposition 4.1. *Suppose that the dynamo bifurcation from the α state is supercritical. Let α_{b} denote the bifurcated 'magnetic' steady state, which is invariant by κ_y, the reflection through the plane Oxz. Then a robust heteroclinic cycle bifurcates, which connects the 'magnetic' states α_{b}, α'_{b} and α''_{b}, where α'_{b} and α''_{b} are the images of α_{b} after rotations by $\frac{1}{2}\pi$ around the Ox- and Oy-axes, respectively. Moreover, an exchange of stability holds between the purely convective heteroclinic cycle and the magnetic one.*

Proof. The proof of this proposition generally relies on extensions of ideas first presented in Chossat *et al.* (1999*b*). The major difference here is the extension from a planar model to a fully spherical model and the inclusion of confirmed convective solutions and their magnetic bifurcations (the bifurcation from the α-convection cell studied in Vivancos *et al.* (1998)) into the model as opposed to an assumed structure of the convection field and an assumed magnetic bifurcation in Chossat *et al.* (1999*b*). Extending the geometry from planar to spherical convection also implies a more complex symmetry, and hence no direct application of the theorem stated in Chossat *et al.* (1999*b*) can be made.

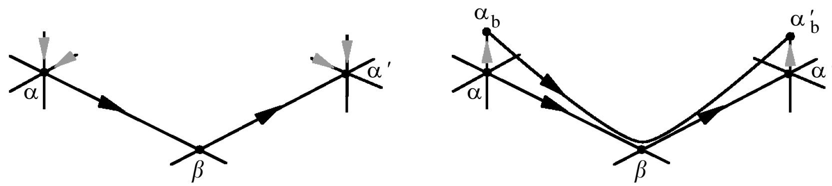

Figure 3. Heteroclinic connections in $\hat{S}$. (*a*) Before dynamo bifurcation. (*b*) After dynamo bifurcation.

First a general remark: asymptotic stability of a robust heteroclinic cycle is determined by the behaviour of the flow in the vicinity of the equilibria. Roughly speaking, a sufficient condition for stability is that, at each equilibrium, the rate of expansion be lower than the rate of contraction along the heteroclinic trajectories, and that in the directions 'transverse' to these heteroclinic orbits, the flow be contracting. Failure of the first condition can lead to 'resonant bifurcation' (see Chossat & Lauterbach 2000). Failure of the other condition occurs when a bifurcation holds for one (or more) equilibrium. This is precisely our situation. We call this phenomenon a 'transverse bifurcation' for the heteroclinic cycle. An example of transverse bifurcation, related to the present one but for a dynamo problem in planar rotating convection, has been studied by Chossat *et al.* (1999*b*).

Let us now analyse the implications of the assumptions in the theorem. As we already noticed, the dynamo bifurcation from the steady state α is similar to the usual steady-state bifurcation with $O(2)$ symmetry: a two-dimensional centre manifold exists in a neighbourhood of α, and on this manifold, the bifurcated solutions form a circle of equilibria (generated by rotations around the vertical axis), each of which is fixed under reflection through a vertical plane. In particular, α_{b} denotes the solution which is fixed under the reflection κ_y. If the bifurcation is supercritical, then there is (on the centre manifold) an exchange of stability between α and the circle of bifurcated solutions. We have not tried to check numerically whether supercriticality was a correct hypothesis; however, it is widely accepted from general considerations about the physical model.

Now let $\hat{S} = V^{\kappa_y}$ be the subspace of vectors in V that are fixed under κ_y. This is a flow-invariant space for the semi-flow defined by the evolution equations of the problem. When restricted to $V_{1,2}$, $\hat{S} = \{x_0, x_{1\mathrm{i}}, y_0, y_{1i}, y_{2\mathrm{i}}\}$. However, as we said in the beginning of this section, we cannot restrict our analysis to a finite-dimensional space or centre manifold. Now, the following states belong to $\hat{S}$: α, α', α_{b} and α'_{b}. This is clear for α and α' from their isotropy subgroups. This is also clear for α_{b} because it is fixed under κ_y by assumption (and we know there exists a bifurcated state from α that has this property). In order to prove that $\alpha'_{\mathrm{b}} \in \hat{S}$, notice that α_{b} is fixed by κ_z. On the other hand, $\kappa_y = \rho_x \kappa_z \rho_x^{-1}$ (ρ_x is the rotation by $\frac{1}{2}\pi$ around Ox). Therefore, $\alpha'_{\mathrm{b}} = \rho_x \alpha_{\mathrm{b}}$ is fixed by κ_y. Next, notice that before the dynamo bifurcation α' is a sink in $\hat{S}$. This follows from our remark that (i) α' is a sink in $\hat{S}$ (see Chossat *et al.* 1999*a*) and (ii) the exchange of stability holds for a supercritical dynamo bifurcation.

Now we consider the unstable manifold W^u of α_{b}. Since the unstable manifold of α is one dimensional and lies in $\hat{S}$ (this is the heteroclinic orbit to β), after bifurcation, W^u is one dimensional and close to the heteroclinic orbit. However, even if a bifurcation has also occurred from β to magnetic states, W^u 'misses' the stable manifold of these states and ends up in a neighbourhood of α'_{b}, where it meets the stable manifold of this state because it is a sink in $\hat{S}$. This fact is a generic property of perturbations of vector fields. To claim in full rigour that it actually occurs in our case, one should, in principle, perform a perturbation analysis of Melnikov type. However, numerical simulations show this is indeed the case (see next section), and we do not need to check this further. Therefore, a robust connection is realized between $\alpha_{\rm b}$ and $\alpha'_{\rm b}$, at least close to the bifurcation. Figure 3 shows a sketch of this phenomenon. By symmetry, this connection defines a heteroclinic cycle connecting magnetic states in the $O(3)$ orbit of $\alpha_{\rm b}$. One can easily check whether this cycle only involves the states $\alpha_{\rm b}$, $\alpha'_{\rm b}$ and $\alpha''_{\rm b}$. The exchange of stability between the initial heteroclinic cycle and this bifurcated heteroclinic cycle can be proved by the same arguments as in Chossat *et al.* (1999*b*) for the so-called 'type-B' homoclinic cycles. ■

Remark 4.2. The bifurcated heteroclinic cycle connects magnetic states, which are all symmetric copies of $\alpha_{\rm b}$. It is usual to call this a *homoclinic cycle*. It is remarkable that a transverse bifurcation from a heteroclinic cycle (even a generalized one) leads to a simple homoclinic cycle. While this is not too surprising for a dynamical system, it has, to our knowledge, not been observed in any discussion of a physical model.

The consequence of this theorem is that when a dynamo mechanism is excited from the flow associated with the state α the magnetic field undergoes an intermittent behaviour, which manifests itself as a long-standing steady state followed by a sudden 'far from equilibrium' excursion and relaxation to a new steady state, which is similar to the initial one after some rotation by 90°. We may ask what would happen if the β state had undergone a dynamo bifurcation and not the α state. In this case, α' is still a sink in $\hat{S}$. Therefore, robust connections from $\beta_{\rm b}$ to α' exist. Also, a robust connection exists from β to $\beta_{\rm b}$ (centre manifold). We therefore have a heteroclinic cycle $\alpha \to \beta \to \beta_{\rm b} \to \alpha'$. However, the rate of expansion from β to $\beta_{\rm b}$ being very weak compared to the one from β (or $\beta_{\rm b}$) to α', it can be shown that for a generic initial condition in a neighbourhood of one of the equilibria the forward trajectory asymptotically converges to the purely convective heteroclinic cycle $\alpha \to \beta \to \alpha'$ (see Chossat *et al.* 1999*b*). This is a case of the 'self-killing' dynamo (Fuchs *et al.* 2001).

5. A low-dimensional model for the dynamo in a spherical shell

Based on the bifurcation studies for mode interaction for convection problems in spherical shells (see § 3) and on the bifurcations in the kinematic dynamo problem from the steady states of the convection problem (see § 2), we can now try to derive a low-dimensional model that incorporates both bifurcation problems and, in this way, leads to the homoclinic magnetic cycles as discussed in § 4. Ideally, one would like to have a degenerate bifurcation from the spherically symmetric static heat conducting state simultaneously to convecting states with and without magnetic components. This would correspond to a codimension-3 bifurcation, where we would have fixed the magnetic Reynolds number $Re_{\rm m}$ and the Rayleigh number Ra at a critical value, together with an aspect ratio η_r for the spherical shell that would allow for the $\ell = 1$ and $\ell = 2$ mode interaction. If this were possible, we could then perform a centre manifold reduction leading to a finite-dimensional system that exactly describes the dynamics in the vicinity of the codimension-3 point. Unfortunately, this does not happen in a physical situation, since, at the onset of convection, the flow speed is zero, leading to a zero magnetic Reynolds number Re_{m}. We therefore try to approximate the full attractor for the magnetohydrodynamic equations by coupling the codimension-2 convective model interaction to magnetic modes in order to achieve a low-order model that is consistent with the bifurcations discussed so far. The major conditions for such a model are as follows.

(i) For $\boldsymbol{B} = \boldsymbol{0}$, the system reduces to (3.2), which describe the evolution of the purely convective components on the eight-dimensional centre manifold of the $\ell = 1, \ell = 2$ interaction.

(ii) There exists a supercritical dynamo bifurcation from the $\boldsymbol{v}_-$ flow (α state) into the $m = 1$ direction consistent with the results in Vivancos *et al.* (1998).

(iii) The system has to respect the symmetries of the problem, i.e. for the non-rotating case, the differential equations have to be equivariant with respect to the group $O(3)$, whereas for the rotating case, the equivariance has to be relative to $C_{\infty\mathrm{h}}$.

(iv) The system has to respect the inversion symmetry T of the magnetic field $\boldsymbol{B} \to -\boldsymbol{B}$.

For simplicity, we will capture the dynamo bifurcation by spherical modes with $\ell = 1$ and denote them by z_0, $z_{\pm 1}$. The resulting equations up to order three are as follows:

$$\left.\begin{aligned}
\dot{x}_0 &= f_0(\mu_1, x, y) + \alpha x_0 \|z\|^2 + \beta z_0(x_0 z_0 - x_1 z_{-1} - x_{-1} z_1),\\
\dot{x}_1 &= f_1(\mu_1, x, y) + \alpha x_1 \|z\|^2 + \beta z_1(x_0 z_0 - x_1 z_{-1} - x_{-1} z_1),\\
\dot{y}_0 &= g_0(\mu_2, x, y) + \gamma(z_0^2 + z_1 z_{-1}) + \delta y_0 \|z\|^2,\\
\dot{y}_1 &= g_1(\mu_2, x, y) + \sqrt{3}\gamma z_0 z_1 + \delta y_1 \|z\|^2,\\
\dot{y}_2 &= g_2(\mu_2, x, y) + \sqrt{3/2}\gamma z_1^2 + \delta y_2 \|z\|^2,\\
\dot{z}_0 &= z_0(\lambda_{\mathrm{b}} + b\|x\|^2 + b'\|y\|^2 + b''\|z\|^2)\\
&\qquad + c(z_0 y_0 - \sqrt{3} Re(y_1 z_{-1})) - c' \Sigma_0^3(z, y, y),\\
\dot{z}_1 &= z_1[\lambda_{\mathrm{b}} + b\|x\|^2 + b'\|y\|^2 + b''\|z\|^2]\\
&\qquad + 0.5c(-y_0 z_1 + \sqrt{3} y_1 z_0 - \sqrt{6} y_2 z_{-1}) - c' \Sigma_1^3(z, y, y).
\end{aligned}\right\}$$

The action of the symmetries on the variables is defined by the type of representation of $O(3)$ in the corresponding space: $\ell = 1$ and $\ell = 2$ for the non-magnetic modes, with the natural action of the antipodal reflection σ, and $\ell = 1$ for the magnetic variables, with the anti-natural action of σ (i.e. σ transforms $B(x)$ into $B(-x)$ instead of $-B(-x)$). Since, however, we also have the symmetry $T : B \to -B$, the action on the magnetic variables of the complete symmetry group $O(3) \times Z_2$ is the same as if $O(3)$ were acting on B through its natural representation with $\ell = 1$.

While this model has the advantage of being consistent under the representations of the symmetries, it is clearly still a severely truncated model of the full magnetohydrodynamic equations and hence subject to truncation errors that may severely affect the dynamics.

For the following simulations, we choose the vector field describing the evolution of the velocity field to be

$$\left.\begin{aligned}
f_0(x,y) &= x_0(\mu_1 - 0.35\|x\|^2 - \|y\|^2)\\
&\quad - 0.3(x_0y_0 - 0.5\sqrt{3}x_1y_{-1}) - 0.4\Sigma_0^3(x,y,y) - \epsilon^2 x_0,\\
f_1(x,y) &= x_1(\mu_1 - 0.35\|x\|^2 - \|y\|^2)\\
&\quad - 0.15(x_1y_0 + \sqrt{3}x_0y_1 - \sqrt{6}x_{-1}y_2)\\
&\quad - 0.4\Sigma_1^3(x,y,y) - 0.8\epsilon^2 x_1 + \epsilon \mathrm{i} x_1,\\
g_0(x,y) &= y_0(\mu_2 - \|x\|^2 - \|y\|^2) + 0.23(x_0^2 + x_1x_{-1})\\
&\quad - 0.08(y_0^2 - y_1y_{-1} - 2y_2y_{-2}) - 0.9\epsilon^2 y_0,\\
g_1(x,y) &= y_1(\mu_2 - \|x\|^2 - \|y\|^2) + 0.23\sqrt{3}x_0x_1\\
&\quad - 0.08(y_0y_1 - \sqrt{6}y_{-1}y_2) - 0.77\epsilon^2 y_1 + 1.11\epsilon \mathrm{i} y_1,\\
g_2(x,y) &= y_2(\mu_2 - \|x\|^2 - \|y\|^2) + 0.23\sqrt{1.5}x_1^2\\
&\quad - 0.08(-2y_0y_2 + 0.5\sqrt{6}y_1^2) - 0.5\epsilon^2 y_2 + 2.22\epsilon \mathrm{i} y_2,\\
\Sigma_0^3(z,y,y) &= z_0(1.5y_0^2 - 2y_1y_{-1} - y_2y_{-2}) - 0.5\sqrt{3}(z_1y_0y_{-1} + z_{-1}y_0y_1)\\
&\quad + 1.5\sqrt{2}(z_1y_{-2}y_1 + z_{-1}y_{-1}y_2),\\
\Sigma_1^3(z,y,y) &= 0.5(-z_1y_1y_{-1} + \sqrt{3}z_0y_0y_1 - 3\sqrt{2}z_0y_{-1}y_2 - 3z_{-1}y_1^2)\\
&\quad + \sqrt{6}z_{-1}y_0y_2 + 2z_1y_{-2}y_2.
\end{aligned}\right\} \tag{5.2}$$

We choose the interaction parameters between magnetic and velocity field to be $\alpha = -0.3$, $\beta = -0.2$, $\delta = -0.4$, $\lambda_\mathrm{b} = -0.2$, $b = 0.18$, $b' = -0.9$ and $b'' = -0.8$. In principle, the parameters α, β, etc., are order-one parameters. In a strict unfolding of a degenerate bifurcation describing a simultaneous bifurcation of the convection and the magnetic problem, these parameters could be calculated numerically. However, since we do not have such a case, we will choose them judiciously. Specifically, we chose the coefficients in the pure convection problem (5.2) to generate the structurally stable heteroclinic cycles discussed in Friedrich & Haken (1986). In addition, several coefficients have a unique relationships to the physical problem and can therefore be restricted. For example, $\lambda_\mathrm{b} < 0$, since it is an eigenvalue of the Laplacian operator. We are also assuming a supercritical dynamo bifurcation from the α or β convection cells (or both). This restricts the coefficients that govern the magnetic bifurcation, notably b', b'', c and c'. Notice that all the interaction terms b, b', b'', as well as α, β and c, are proportional to the magnetic diffusivity. The parameter ϵ reflects the influence of rotation on the convection problem via Coriolis forces. For our simulations, we vary the following parameters: μ_1, μ_2 describing the convective bifurcation for the $\ell = 1$ and the $\ell = 2$ mode; c and c' controlling the magnetic bifurcation; γ describing the feedback of the magnetic system onto the convection problem; and ϵ controlling the rotation of the system.

6. Simulations of homoclinic and heteroclinic dynamos with rotation

Even after fixing most of the parameters in (5.1), we cannot explore the remaining parameter space completely. We will therefore focus on a simulation that illustrates

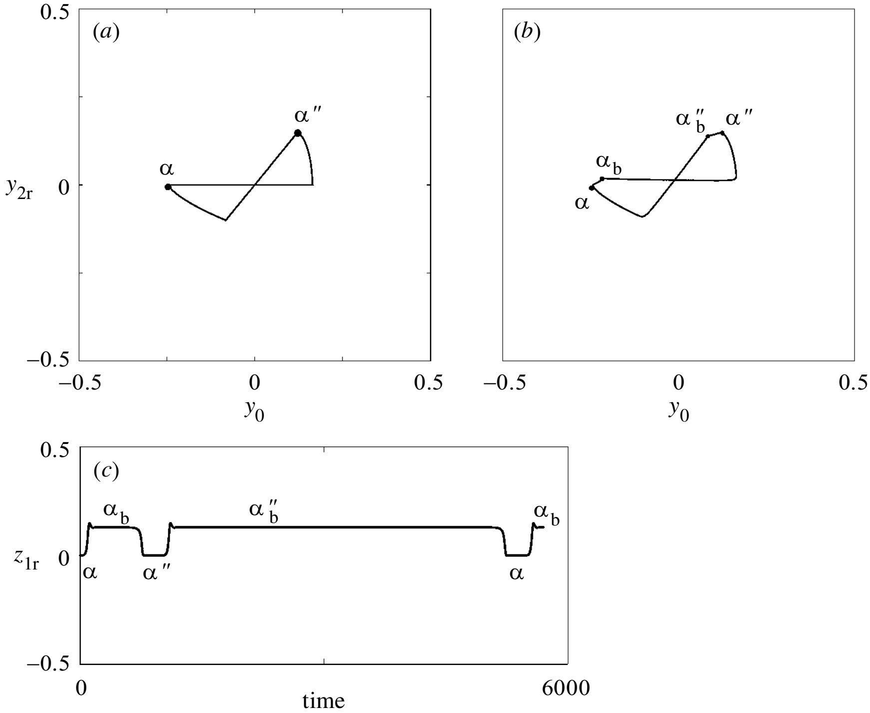

Figure 4. Projection of a simulation of (5.1) into the (y_0, y_{2r})-plane. (*a*) With initial conditions in the convection system. (*b*) With general initial conditions. (*c*) Time-series of the magnetic component z_{1r} for the phase portrait in (*b*). Dots in (*a*) and (*b*) correspond to fixed points.

the bifurcation behaviour of the convective heteroclinic cycle to magnetic cycles. For $\mu_1 = 0.033$, $\mu_2 = 0.04$, we have already shown in Armbruster & Chossat (1991) that there exists a heteroclinic cycle in the non-rotating convective system of the form $\alpha \to \beta \to \alpha' \to \beta' \to$ etc. Figure 4*a* shows the projection of this heteroclinic cycle into the (y_0, y_{2r})-plane.

By choosing $c = 2.7$, $c' = 8$, we arrange that the β fixed point is magnetically stable, while the α fixed point is unstable into the z_1-direction. With no rotation ($\epsilon = 0$) and weak feedback ($\gamma = 0.05$), our simulation shows a picture exactly as predicted in figure 3: the α fixed point bifurcates to a circle of fixed points in z_1 and initial conditions choose the specific location on that circle that a particular trajectory goes to. That circle is unstable in the x_0-direction. A trajectory leaving from there comes close to the β fixed point, but does not stay very long before it leaves to α'. Figure 4*b* shows the same (y_0, y_{2r}) projection as in figure 4*a*. We see that there are additional fixed points close to the α and α'' points. Figure 4*c* shows a time-series for the z_{1r} component, which confirms that those additional fixed points are magnetic points. In addition, figure 4*c* is a transient that shows that the magnetic homoclinic cycle is asymptotically stable and, as a result, the trajectory spends longer and longer time at the fixed points.

Simulating the same situation (we only change μ_1 to 0.025 to speed up the convective dynamics) with rotation leads to rotating waves for all states that do not have the same symmetry axis as the rotation axis. We discuss theoretical details

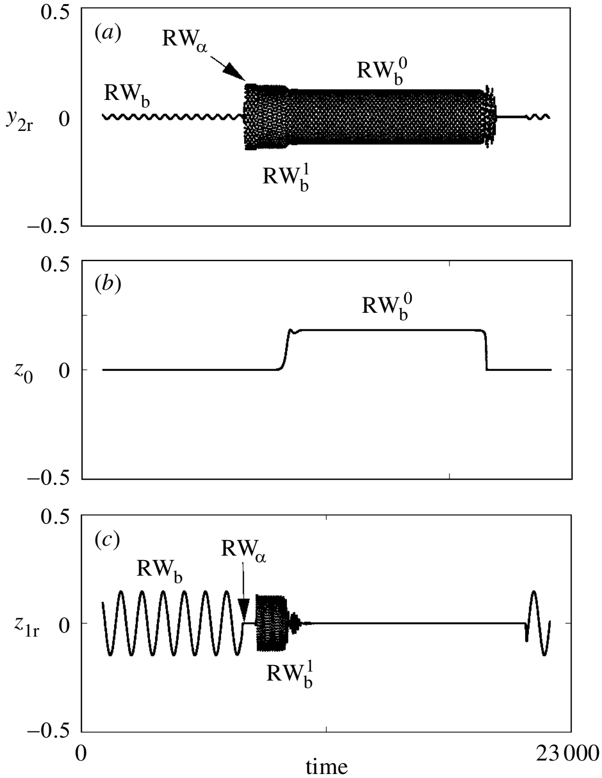

Figure 5. Time-series of (a) the velocity component y_{2r}, (b) the magnetic component z_0 and (c) the magnetic component z_{1r}.

of this simulation in the next section and only describe the simulation here. The bifurcation from α now leads to a rotating magnetic wave in the z_1-plane. The pure convective states α' and α'' are now part of a rotating wave in the y_2-plane and they are bifurcating to a rotating wave with a z_1 component. Phase plane projections of resulting heteroclinic cycles are not very revealing. However, multiple time-series for several variables allow us to recover the structure of the heteroclinic cycle. Parts (a), (b) and (c) of figure 5 show the time signals for y_{2r}, z_0 and z_{1r} over a time-interval of one complete cycle, respectively. We start out as a slowly rotating almost purely magnetic wave in z_1 near the α fixed point ($\mathrm{RW_b}$). This becomes unstable (as before, in the x_0-direction) and the trajectory moves to a fast rotating wave near the α' fixed point ($\mathrm{RW_b^1}$). That wave consists of an $\ell = 2$ component in the velocity field and an $m = 1$ component in the magnetic field. The frequency of rotation has increased dramatically. In addition, a projection into the complex z_1-plane shows that the two rotating waves are rotating in opposite directions. All of the states so far can be understood as simple drifts of the non-rotating structures. However, once the fast rotating magnetic wave becomes unstable, the trajectory moves to a new state ($\mathrm{RW_b^0}$), characterized by a steady magnetic component z_0 and a rotating velocity component (see figure 5b). When it becomes unstable, the orbit returns to the magnetic wave at the α fixed point.

7. A geometrical description of the heteroclinic cycle in the case with rotation

The simulations discussed in the previous section indicated that, even when rotation is present, a heteroclinic behaviour can persist between bifurcated magnetic and non-magnetic states. The aim of this section is to give compelling theoretical reasons that, at least for the low-dimensional model, this is indeed the case.

First, let us precisely state the structure of the heteroclinic cycle that occurs in the rotating system in the absence of dynamo. The following facts were proved in Chossat *et al.* (1999*a*) for ϵ close enough to 0. We recall that when $\epsilon \neq 0$ the symmetry group of the system is reduced to $C_{\infty\mathrm{h}}$, the direct product of the rotation group C_∞ around Oz and of the antipodal reflection group $\{\sigma, \mathrm{I}\}$.

(i) The $\alpha \rightarrow \beta$ connection in P_0 persists (P_0 is still an invariant plane).

(ii) Recall that RW_α and RW_β denote the rotating waves that are induced by the rotation from the steady states α' and β' (or, equivalently, from α'' and β''). Then connections $\beta \rightarrow \mathrm{RW}_\alpha$ and $\mathrm{RW}_\beta \rightarrow \alpha$ persist in the space $\{y_0, y_2, \bar{y}_2\} = V^{C_{2\mathrm{h}}}$, where $C_{2\mathrm{h}}$ is generated by ρ_z^2 (rotation by π around Oz) and by σ.

(iii) The two-dimensional $SO(3)$ orbit of α, which we write O_α, is perturbed, when rotation sets in, to a flow-invariant manifold O_α^ϵ, which consists of the steady state α, the rotating wave RW_α and a 2-manifold of heteroclinic orbits from α to RW_α (in forward time). The same thing happens for the β state.

(iv) α is a sink in the still flow-invariant space S and the unstable manifold of RW_α is included into the stable manifold of α in S.

The persisting heteroclinic cycle thus realizes the connections $\alpha \rightarrow \beta \rightarrow \mathrm{RW}_\alpha \rightarrow \alpha$. We now assume that a dynamo bifurcation has occurred from α (but not from β, in order to simplify the analysis). We first describe the dynamics that is induced when $\epsilon \neq 0$ on the $SO(3)$ orbit of the magnetic steady state α_b, which we will call O_{α_b} for convenience. This orbit O_{α_b} is a three-dimensional compact manifold that has bifurcated out of the two-dimensional O_α, and which we will assume to be normally hyperbolic, a natural hypothesis unless accidental degeneracies occur. As rotation is set in and as long as ϵ is close enough to 0, O_{α_b} is perturbed to an invariant manifold $O_{\alpha_\mathrm{b}}^\epsilon$ on which a rotation-induced drift flow appears.

Proposition 7.1. *Under the above assumptions, there exists a number* $\epsilon_0 > 0$, *depending on the parameters of the problem, such that if* $0 < |\epsilon| < \epsilon_0$ *a flow-invariant manifold* $O_{\alpha_\mathrm{b}}^\epsilon$ *persists near* O_{α_b}. *On this manifold, three rotating waves* RW_b, RW_b^1 *and* RW_b^0 *appear, such that*

(1) RW_b *is close to* α *in the space* $\mathrm{Fix}(\kappa_z)$;

(2) RW_b^1 *is close to* RW_α *in* $\mathrm{Fix}(\kappa_z)$;

(3) RW_b^0 *is close to* RW_α *in* $\mathrm{Fix}(T\kappa_z)$. *Moreover,* RW_b *is a repellor along the manifold* $O_{\alpha_\mathrm{b}}^\epsilon$.

Proof. The largest subgroup of $O(3)$ that acts in $\mathrm{Fix}(\kappa_z)$ is the normalizer of the subgroup $\{\kappa_z, \mathrm{I}\}$ in $O(3)$ (this follows from a general property of linear group actions (see Chossat & Lauterbach 2000)). This normalizer is $D_{\infty\mathrm{h}}$. Therefore, the intersection of O_{α_b} with $\mathrm{Fix}(\kappa_z)$, if not empty, consists of a union of isolated circles. Normal hyperbolicity implies that these circles persist as invariant sets for the dynamics, as long as ϵ is close enough to 0. Moreover, the dynamics on these circles is a uniform drift (rotating wave), due to their C_∞ invariance. Now, we can identify two points on O_{α_b} that belong to $\mathrm{Fix}(\kappa_z)$: α_b and α'_b. Indeed, we know that α_b is fixed by κ_z and by κ_y, and $\alpha'_\mathrm{b} = \rho_x \alpha_\mathrm{b}$. Then we just observe that $\kappa_z = \rho_x^{-1}\kappa_y\rho_x$. Points (1) and (2) follow. To prove (3), notice that α_b is transformed to $-\alpha_\mathrm{b}$ under the reflection κ_x. Indeed, κ_x is the product of σ by $\kappa_y\kappa_z$, and we know that $\sigma\alpha_\mathrm{b} = -\alpha_\mathrm{b}$, while κ_y and κ_z keep α_b fixed. Now, $\rho_y\kappa_x\rho_y^{-1} = \kappa_z$. Since $\alpha''_\mathrm{b} = \rho_y\alpha_\mathrm{b}$, we conclude that $\kappa_z\alpha''_\mathrm{b} = -\alpha''_\mathrm{b}$ and therefore $\alpha''_\mathrm{b} \in \mathrm{Fix}(T\kappa_z)$. The rest of the argument is similar to that of (1), (2). Finally, $\mathrm{RW_b}$ is unstable along $O^\epsilon_{\alpha_\mathrm{b}}$ because α has this property along O^ϵ_α. ∎

We can now list a sequence of robust heteroclinic connections that, if they exist, define a heteroclinic cycle involving the 'magnetic' rotating waves of the above proposition. Their existence is numerically attested on the low-dimensional problem of § 5. We recall that ρ_z^2 is the rotation by π around Oz.

(i) $\alpha \to \mathrm{RW_b}$. Exists because $\mathrm{RW_b}$ belongs to the centre manifold of α (and the exchange of stability holds on this manifold).

(ii) $\mathrm{RW_b} \to \mathrm{RW_b^1}$. Exists in $\mathrm{Fix}(T\rho_z^2)$. Indeed, both rotating waves belong to this invariant subspace, which also contains the connections of purely convective states $\alpha \to \beta \to \mathrm{RW}_\alpha$, the latter being stable in that space intersected with $\mathrm{Fix}(T)$. The same type of perturbation argument as in the proof of proposition 4.1 allows us to conclude the existence of this connection.

(iii) $\mathrm{RW_b^1} \to \mathrm{RW_b^0}$. Exists if one assumes that $\mathrm{RW_b^1}$ is repelling along the invariant manifold $O^\epsilon_{\alpha_\mathrm{b}}$ while $\mathrm{RW_b^0}$ is attracting, and there is no other attracting rotating wave on $O^\epsilon_{\alpha_\mathrm{b}}$. This is a natural hypothesis, which is corroborated by the numerical simulations.

(iv) $\mathrm{RW_b^0} \to \alpha$. Exists in $\mathrm{Fix}(T\kappa_z)$. This follows from the fact that α is a sink in $\mathrm{Fix}(\kappa_z)$ and a two-dimensional family of orbits connect RW_α to α in this space (see the description of the purely convective heteroclinic cycle with rotation in the beginning of this section). Again, a perturbation argument leads to the conclusion.

This sequence of connections corresponds precisely to the observed time-series of the non-magnetic and magnetic amplitudes of the 11-dimensional model, as reported in § 6.

This heteroclinic cycle is of a more complex structure than the one found in the non-rotating case (proposition 4.1). Indeed, it involves not only the magnetic states, but also a non-magnetic one (α). The situation is the following. The unstable manifold of α has dimension three and is connected not only to the bifurcated magnetic rotating wave $\mathrm{RW_b}$, but also to the steady state β. In fact, the 'purely convective' heteroclinic cycle is still there and is part of the full object, but it is not automatically unstable, unlike what happens in the non-rotating case. The asymptotic behaviour near such objects has been investigated in Ashwin & Chossat (1998). If the complete heteroclinic cycle is an attractor, the asymptotic dynamics nearby can tend towards a 'sub-cycle' such as the purely convective one. This depends on the relative rates of expansion and contraction at α and at the rotating waves, and a precise analysis would be very tedious. Numerically, it has been observed on the 11-dimensional model that if ϵ is too large the system relaxes to the non-magnetic heteroclinic cycle, another example of the 'self-killing' dynamo (Fuchs *et al.* 2001) (where now the rate of rotation is responsible for this phenomenon).

8. Discussion

We have shown that a structurally stable homoclinic cycle will exist in the full magnetohydrodynamic equations without rotation. This cycle will connect two rotated copies of the same steady states. Each steady state consists of two almost axisymmetric parts: a fluid flow generated by convection and dominated by a spherical harmonic of the form Y_{20} and a magnetic dipole oriented orthogonal to the symmetry axis of the convective flow. It is worth emphasizing that the system has an *almost* axisymmetric flow and is *almost* a dipole, but not exactly. In addition, we have shown the possibility of a structurally stable heteroclinic cycle for the full magnetohydrodynamic equations with rotation and confirmed its existence in a low-dimensional model of convection-driven magnetic fields. In this case, the rotation leads to the generation of rotating waves that are connected via a heteroclinic cycle. Specifically for the 11-dimensional model, we find that the following states are visited by a trajectory.

(I) The convection flow, due to its dominant axisymmetry, changes only weakly under rotation and has a small drifting component. However, the magnetic dipole, having its symmetry axis orthogonal to the rotation axis, starts to drift along the rotation axis.

(II) The convection flow is axisymmetric, but its symmetry axis is orthogonal to the rotation axis. Hence the whole flow also drifts along the rotation axis. The associated magnetic field is composed equally of the rotating dipole orthogonal to the rotation axis and another dipole component that does not drift, since it is pointing along the rotation axis.

(III) The rotating part of the magnetic field decays and, as a result, the magnetic dipole stops rotating and points along the rotation axis.

Time-series of the magnetic field for these trajectories give important insight into the dipolar reversals. The general properties of SSHCs allow us to change the relative time that a trajectory spends near the various metastable states involved in the heteroclinic cycle. Roughly speaking, for any saddle with a one-dimensional unstable manifold in the cycle, the time spent near that saddle depends on the relative magnitude of the unstable eigenvalue λ_u and the least stable eigenvalue $-\lambda_s$. If $\lambda_u/\lambda_s > 1$, then the trajectory will spend only a short time near that saddle point. If $\lambda_u/\lambda_s < 1$, then a trajectory will spend a long time near the saddle until noise will drive it around the cycle. Hence, by changing the eigenvalues at the states I–III, we can generate a trajectory that spends most of its time in a dipolar state, aligned with the rotation axis, and then does an excursion through all the other states before coming back to state III.

We can also note that a dipole that is directed in any other direction than the rotation axis will generically rotate. This is a result for general symmetric bifurcation theory and is not specific to the states that we have discussed here. Hence we expect that, with the inclusion of higher-order modes in the magnetic field ($\ell = 2, 3, \dots$), the state-III dipole will acquire a slight tilt and therefore start to drift around the rotation axis, as does the Earth's magnetic field.

Finally, because of the gauge symmetry T in the magnetic field, any solution with a magnetic field of $\boldsymbol{B}$ for these equations will have a symmetric solution of $-\boldsymbol{B}$. Reversals can happen if, along the way of a trajectory, we come to a state that is invariant under that symmetry such that small noise perturbations can move the trajectory from a $+\boldsymbol{B}$ state to a $-\boldsymbol{B}$ state. Such states include the purely convective states, when the magnetic field has died to zero momentarily, but also a state I, whose magnetic component is a rotating dipole. As a result, long time-series of the SSHC show aperiodic changes of the direction of the dipole in the state III.

It is worth noting that the three-dimensional simulations of Sarson & Jones (1999) drive their magnetic field with a convective flow that is very similar to the flow associated with our states II and III. They present a dipolar reversal. However, their simulation does not resemble the intermittent time-series of the geodynamo. They restrict themselves to two convection modes for a highly supercritical and fast rotating flow. It would be interesting to see whether a simulation that includes all the modes related through the spherical symmetry show a more intermittent time-series.

Our analysis has two important shortcomings inherent in most rigorous results from bifurcation theory. Firstly, many of our arguments rely on the existence of a centre manifold and hence are local to the magnetic bifurcation. Secondly, the rigorous proofs depend on the continuation of the convective SSHCs for an $O(3)$ symmetric (i.e. non-rotating) problem to a $C_{\infty\mathrm{h}}$-symmetric (i.e. rotating) problem. These continuation arguments are again perturbative in nature and hence strictly valid only asymptotically for a rotation rate going to zero. However, the numerical simulations in § 6 use rotation rates of the order of magnitude of the bifurcation parameters that reflect the Rayleigh or magnetic Reynolds numbers. In addition, the rotations generated are on a time-scale much shorter than the time-scales of the transition times between heteroclinic orbits reflecting a non-trivial rotation. As with all bifurcation analysis, there is hope, substantiated by lots of experience, that the validity of its analysis is much larger than it can be proven mathematically. With that, we hope to point to a possible and arguably likely mathematical mechanism for the reversal of the geodynamo.

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