Noisy heteroclinic networks

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The influence of small noise on the dynamics of heteroclinic networks is studied, with a particular focus on noise-induced switching between cycles in the network. Three different types of switching are found, depending on the details of the underlying deterministic dynamics: random switching between the heteroclinic cycles determined by the linear dynamics near one of the saddle points, noise induced stability of a cycle, and intermittent switching between cycles. All three responses are explained by examining the size of the stable and unstable eigenvalues at the equilibria. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1539951]

An asymptotically stable heteroclinic cycle is an attractor of a system of nonlinear differential equations that consists of a finite number of equilibria which are cyclically connected. Such heteroclinic cycles can occur in a structurally stable way in systems with symmetry and in evolutionary game theory. A heteroclinic network consists of two or more heteroclinic cycles that share some but not all of the equilibria and connections in the corresponding cycles. Within such a network there is competition between the different heteroclinic cycles that make up the network. It has been shown that the eventual fate of a solution trajectory near a heteroclinic network depends crucially on whether the trajectory visits extremely small cusp-like regions in phase space. Following this it has been suggested that the addition of noise will smear out the deterministic behavior in phase space and thus simplify the resulting behavior. The present study refutes a simplistic noise influence and shows that the dynamics is determined by the intricacies of the interplay between noise and deterministic dynamics.

I. INTRODUCTION

The study of the effect of noise on dynamical systems is critical, since the physical situations that they model will never be completely free from random perturbations. Of particular interest to us is the situation where very small noise can make a big difference in the observable behavior of a system, i.e., the system acts as a noise "amplifier." Some examples of this phenomena include stochastic resonance,[1,2] and other studies of systems that are near criticality.[3] As part of the development of a fluid turbulence model,[4] structurally stable heteroclinic cycles emerged as another dynamic feature that can be sensitive to small additive noise. These cycles, if attracting, maintain the same orbit in the presence of tiny noise, but their period is profoundly affected; in the absence of noise the time to complete one cycle will tend to infinity, while the addition of small noise creates a well defined mean period that depends in part on the noise level. The scaling properties of this noise-induced time scale are derived in Ref. 5. Further work on similar cycles made up of more than two fixed points was carried out in Ref. 6, specifically addressing the case when tiny noise could create an observable spread in the trajectories around the cycle. Here we study the effect of small noise on a network of heteroclinic cycles, such as that described in Ref. 7, where the trajectory is presented with a "choice" of cycles at one saddle in the network. Characterizing the noise-induced switching between cycles leads us to understand the interplay between the noise and the deterministic properties of the cycles.

Heteroclinic cycles are characterized by the fact that due to some special structure of the vector field (e.g., equivariance under a symmetry group) there are invariant subspaces in which saddle-saddle connections exist, with the connections being structurally stable with respect to perturbations that preserve the invariant subspaces.[8] The prototypical example in $\mathbf{R}^3$ (Ref. 9) (hereafter called the Busse cycle) has three saddle points on the coordinate axes with one-dimensional unstable manifolds [see Fig. 1(a)]. In this case, all coordinate planes and all axes are invariant subspaces. Restricting the dynamics to any coordinate plane, two of the three saddles in the cycle lie in that plane but one of these looks like a sink in the plane. If a connection between the two equilibria exists within the coordinate plane, then it is a saddle-sink connection in that plane, and is thus robust to perturbations that preserve the invariance of the plane. If such saddle-sink connections exist in all three coordinate planes, then the union of these three connections comprises a structurally stable heteroclinic cycle. The heteroclinic cycle may also be asymptotically stable, in which case solutions will approach the cycle as $t \to \infty$.

Coexisting heteroclinic cycles that have a common heteroclinic connection and that are not related by a symmetry

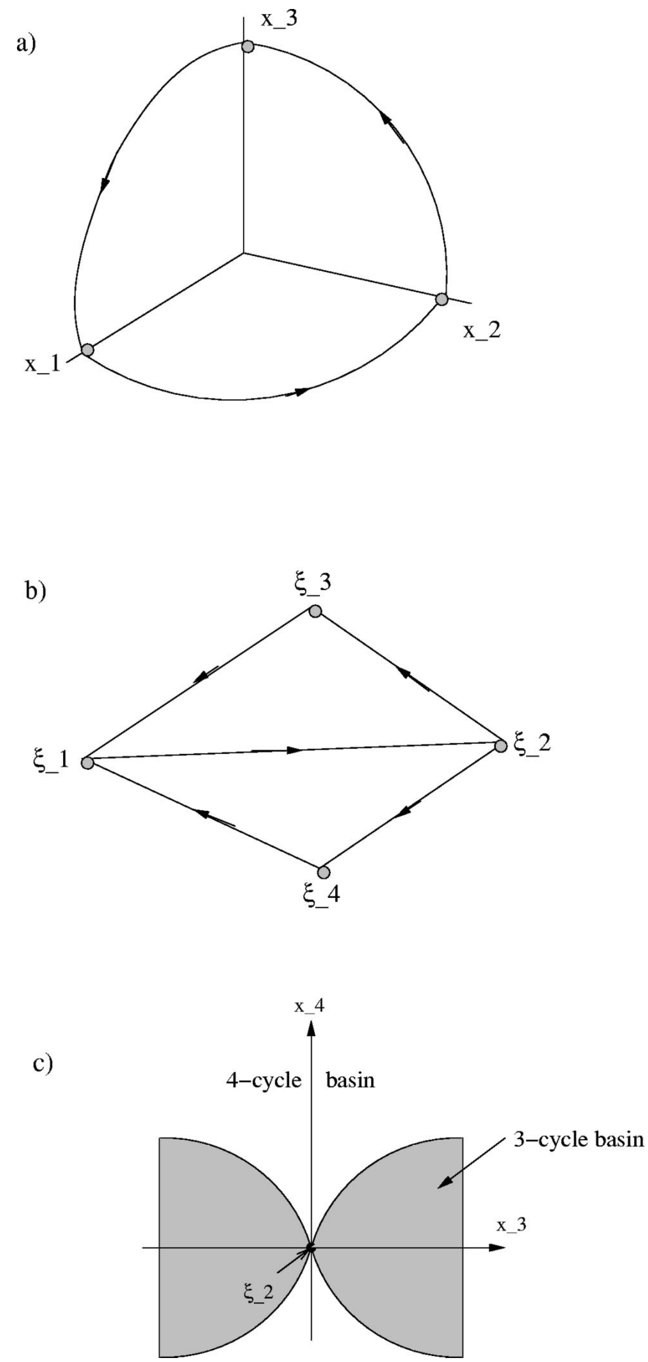

FIG. 1. (a) Busse cycle. (b) Cartoon of a heteroclinic network. (c) Schematic basins of attraction of the three- and four-cycle in the network in (b). Orbits leaving ξ_2 in the direction of ξ_4 have x_3, x_4 coordinates lying in the cusp-shaped region, i.e., orbits have passed through a cuspoidal region abutting the heteroclinic connection from ξ_1 to ξ_2.

are typically referred to as heteroclinic networks. They can occur in a structurally stable way in systems with symmetry[7] or in the replicator equations of population dynamics.[10]

Kirk and Silber[7] discussed a case of two competing heteroclinic cycles, consisting of a system of ordinary differential equations in $\mathbf{R}^4$ with $\mathbf{Z}_2^4$ symmetry. A typical example of the type of system they studied is

$$\begin{aligned}
\dot{x}_1 &= x_1(1-x_1^2-x_2^2-x_3^2-x_4^2) - c_{21}(x_2^2 x_1) + e_{31}(x_3^2 x_1) \\
&\quad + e_{41}(x_4^2 x_1), \\
\dot{x}_2 &= x_2(1-x_1^2-x_2^2-x_3^2-x_4^2) + e_{12}(x_1^2 x_2) - c_{32}(x_3^2 x_2) \\
&\quad - c_{42}(x_4^2 x_2), \\
\dot{x}_3 &= x_3(1-x_1^2-x_2^2-x_3^2-x_4^2) - c_{13}(x_1^2 x_3) + e_{23}(x_2^2 x_3) \\
&\quad - c_{43}(x_4^2 x_3), \\
\dot{x}_4 &= x_4(1-x_1^2-x_2^2-x_3^2-x_4^2) - c_{14}(x_1^2 x_4) + e_{24}(x_2^2 x_4) \\
&\quad - c_{34}(x_3^2 x_4),
\end{aligned} \tag{1}$$

where the c_{ij} and the e_{ij} are positive constants. The system has four equilibria $\xi_1, \xi_2, \xi_3, \xi_4$ on the coordinate axes at $x_i = 1$ and has two competing heteroclinic cycles $\xi_1 \to \xi_2 \to \xi_3 \to \xi_1$ and $\xi_1 \to \xi_2 \to \xi_4 \to \xi_1$ called the three-cycle and the four-cycle, respectively [see Fig. 1(b)]. The two cycles are structurally stable with respect to perturbations that preserve the symmetries of the system. The cycles share a common connection $\xi_1 \to \xi_2$. The equilibrium ξ_2 has a two dimensional unstable manifold containing a continuum of connections to the three-cycle and the four-cycle. The coefficient c_{ij} represents the stable eigenvalue at the equilibrium ξ_i in the x_j direction. The coefficients e_{ij} are the corresponding unstable eigenvalues. Without loss of generality, we assume that the unstable eigenvalues at ξ_2 satisfy $e_{23} > e_{24}$ and hence orbits leaving ξ_2 in the direction of ξ_4 pass through a cuspoidal region abutting the heteroclinic connection from $\xi_1 \to \xi_2$. [See Fig. 1(c).] Kirk and Silber[7] analyze the stability properties of the competing heteroclinic cycles. Specifically they find cases where neither cycle is asymptotically stable but one or both have strong attractivity properties. They further show that switching between cycles can occur in only one direction: a trajectory may start near one cycle and then switch to the other cycle but thereafter may not switch back to the original cycle. They speculate that (i) the influence of noise will lead to orbits switching randomly between the two competing cycles and (ii) the noise will have a large influence on trajectories that pass through the narrow cuspoidal region as they get attracted to the four-cycle. In this paper we show that the influence of noise is in fact more complicated and is determined by intricacies of the interplay between noise and the deterministic dynamics.

As described in Refs. 5 and 6, we are able to model the effect of noise on the system by splitting the phase space into local regions near the saddle points and global regions containing the heteroclinic connections between the saddle points; this splitting is achieved by defining for each saddle point a small box centered on the saddle, with the sides of each box parallel to the coordinate planes. We assume that the noise is small enough to be negligible outside the boxes. In particular, since the vector field is of $O(1)$ on every connecting arc, and we consider noise of rms smaller than 10^{-4}, we assume the dynamics is influenced by the noise only near the fixed points. In Ref. 6 a study is made of the probability distribution of solutions as they pass through the sides of the boxes, either entering or exiting the box-neighborhood of a saddle. Certain estimates of the mean and variance of this distribution are calculated and this information is referred to in our work here.

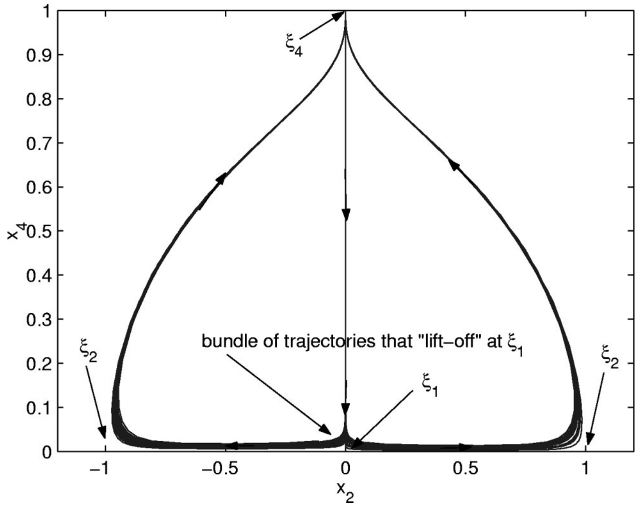

FIG. 2. Busse cycle with added noise ($\epsilon = 10^{-6}$) illustrating liftoff at saddle point ξ_1.

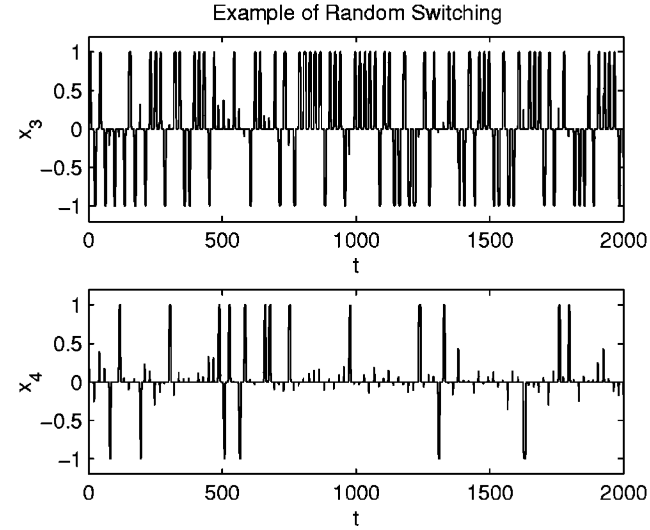

FIG. 3. Time series for $x_3(t)$ and $x_4(t)$ for $(c_{13}, c_{14}, c_{21}, c_{32}, c_{34}, c_{42}, c_{43}, e_{12}, e_{23}, e_{24}, e_{31}, e_{41}) = (4.2, 4.2, 4.3, 4.4, 4.4, 4.4, 4.4, 1.9, 2.5, 2.2, 2.0, 2.0)$. The rms noise level is $\epsilon = 10^{-6}$.

To describe our results about switching we need the following definition.

Definition 1:[11] *Consider a saddle point with eigenvalues* $\lambda_{s_1} < \lambda_{s_2} < \ldots < \lambda_{s_n} < 0$ *and* $\lambda_u > 0$. *The* saddle quantity *for this equilibrium is* $\sigma = \lambda_u + \lambda_{s_n}$.

We have shown in Ref. 6 that the probability distribution of trajectories moving past a saddle-type equilibrium in two dimensions in the presence of noise depends on the eigenvalues of the saddle. For an incoming normal distribution with zero mean the leading order of the mean μ_{exit} of the outgoing distribution scales as $\epsilon^{|\lambda_s|/\lambda_u}$ where ϵ is the rms of the noise process, and λ_u and λ_s are respectively the unstable and stable eigenvalues of the saddle. The corresponding leading order of the standard deviation of the exit distribution is given as

$$\sigma_{\text{exit}} \propto \sqrt{c_1 \epsilon^2 + c_2 \epsilon^{2|\lambda_s|/\lambda_u}} \tag{2}$$

for some positive constants c_1 and c_2. We showed that for a saddle with positive saddle quantity, in the limit $\epsilon \to 0$, $\mu_{\text{exit}} > \sigma_{\text{exit}}$ and the outgoing probability distribution shows a *liftoff* from the coordinate axis, i.e., the bundle of trajectories making up the distribution is lifted away from the noise-free heteroclinic connection. This idea generalizes to liftoff near saddle points in more than two dimensions. Figure 2 shows an example of liftoff for the Busse cycle with additive noise. See Ref. 6 for more details.

In this paper we will show that the heteroclinic network (1) can respond to small additive noise with three different types of switching, depending on the details of the underlying deterministic dynamics:

(1) Orbits switch randomly between the heteroclinic cycles as conjectured in Ref. 7, i.e., the noise induces switching between cycles, with the switching rate essentially independent of the history of the orbit.
(2) No switching between cycles is observed, at least for very long times and apart from initial transients. This case can occur when trajectories are deterministically attracted to either of the heteroclinic cycles; the noise may reinforce the deterministic dynamics (i.e., result in the deterministically preferred cycle being observed) or may reverse the deterministic preference (i.e., pick out the cycle not observed in the deterministic dynamics).
(3) Orbits exhibit intermittent switching between the cycles.

We will explain all three responses by focusing on the size of the stable and unstable eigenvalues at the equilibria. We show that the value of the saddle quantity for the equilibrium ξ_1 and the interaction between the deterministic dynamics and any liftoff near the ξ_1 saddle are the key factors in distinguishing the different cases. Furthermore, in the first of the three cases described above, we are able to make predictions of the switching frequency based only the local dynamics at the ξ_2 saddle point.

We note that the time scale effect, observed in Ref. 5 and mentioned above, is also observed in the heteroclinic networks studied here. In particular, see Figs. 3–5, where the variation in period of the cycle with noise level is apparent. This phenomenon, however, will not enter into our calculation of switching probabilities.

II. CASE STUDIES

To illustrate the three distinct ways that the network can react to small additive noise we plot the x_3 and x_4 time series for three different cases (three different choices of the coefficients c_{ij} and e_{ij}) of system (1) with Gaussian white noise added to each coordinate, during the entire simulation, not just when the trajectory is near a saddle point. The simulations were performed with a noise-adapted second order Runge–Kutta scheme.[12] Each case shows the time series for $x_3(t)$ and $x_4(t)$, including the transient, obtained by starting from a single initial condition. Note that the rms noise level used in the examples ranges from $\epsilon = 10^{-6}$ to $\epsilon = 10^{-4}$, and the average time to complete one circuit of a cycle varies between the different cases, owing to the variation in noise

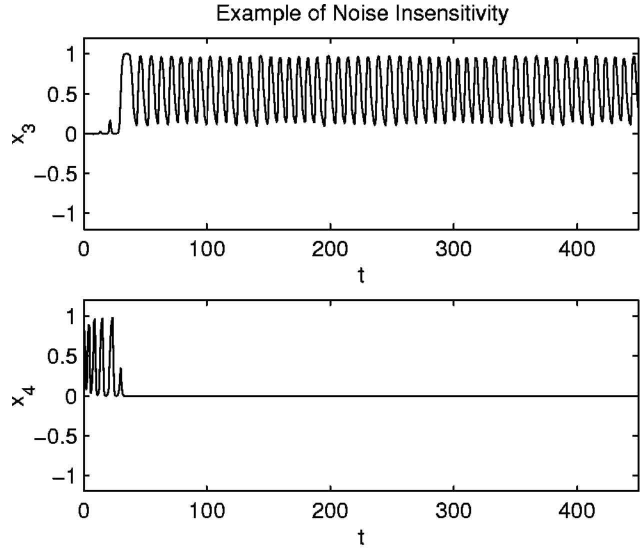

FIG. 4. Time series for $x_3(t)$ and $x_4(t)$ for $(c_{13}, c_{14}, c_{21}, c_{32}, c_{34}, c_{42}, c_{43}, e_{12}, e_{23}, e_{24}, e_{31}, e_{41}) = (0.5,3.3,4.3,4.9,3.8,3.7,3.0,3.5,2.5,2.0,1.0,4.8)$. The rms noise level is $\epsilon = 10^{-5}$.

level and parameters. The mechanisms causing the different types of behavior are described later in the paper.

In describing our results we use the following definition:

Definition 2:[13] *We call a flow invariant set X essentially asymptotically stable (eas) if there exists a set C such that given any number* $a \in (0,1)$ *and any neighborhood U of X there is an open neighborhood* $V \subset U$ *of X such that*

(i) *all trajectories started in* $V \setminus C$ *remain in U and are asymptotic to X and*

(ii) $\mu(V \setminus C)/\mu(V) > a$ *where* μ *is the Lebesgue measure.*

Typically, if a heteroclinic cycle is eas, then all initial conditions in a neighborhood of the cycle outside a cusp-shaped region are attracted to the cycle.

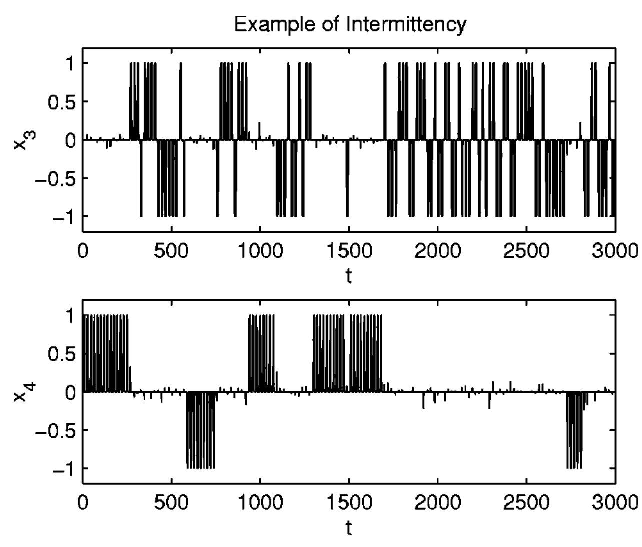

FIG. 5. Time series for $x_3(t)$ and $x_4(t)$ for $(c_{13}, c_{14}, c_{21}, c_{32}, c_{34}, c_{42}, c_{43}, e_{12}, e_{23}, e_{24}, e_{31}, e_{41}) = (6.2,1.0,7.3,2.4,12.7,5.7,5.0,1.5,2.5,2.0,3.0,4.8)$. The rms noise level is $\epsilon = 10^{-7}$.

The first case (Fig. 3) demonstrates the situation where the addition of noise causes random switching between the cycles in the network, with the underlying dynamics seemingly irrelevant in determining the pattern of switching. In the corresponding deterministic system, the three-cycle is eas, while the four-cycle is not eas but attracts a significant set of initial conditions. The saddle point at ξ_1 has negative saddle quantity. Noise causes the trajectory to land mostly in the basin of attraction of the three-cycle, but occasionally in the cusp-shaped part of the basin of attraction of the four-cycle. Every time the trajectory passes near the saddle at ξ_2 it can be affected by noise in this way, and we see apparently random switching between excursions around each of the cycles, with the three-cycle being traversed most frequently.

In the second case (Fig. 4) we show a situation where noise does not appear to have any long-term effect on the pattern of visits to the two cycles. In the underlying deterministic system, the three-cycle is eas and in fact attracts almost all trajectories started near the network. The simulation is started on the four-cycle, but the noise soon pushes the trajectory into the basin of attraction of the three-cycle, where it remains for a very long time.

Finally we show a case where switching between the cycles takes on an intermittent aspect, creating the effect of "blocks" of passages close to the three-cycle alternating with blocks of passages close to the four-cycle. See Fig. 5. In the underlying deterministic system, both cycles attract large sets of initial conditions. In the noisy system, a trajectory started near one of the cycles persists near that cycle until it passes sufficiently close to the basin boundary that noise pushes it into the basin of attraction of the other cycle. The blocking is obvious for $x_4(t)$, where the trajectory does not cross the $x_4 = 0$ axis, but we say there is a block of three-cycles in between the four-cycle blocks, even though $x_3(t)$ changes sign.

In the next section we discuss how these different cases arise and how to determine which will occur upon addition of small additive noise to a heteroclinic network.

III. NOISE AND DYNAMICS

The three cases above correspond to the network with qualitatively different spectra for the saddle points ξ_1 and ξ_2. We first give a heuristic explanation of how different spectrums result in different noise-induced dynamics, then provide details in the following subsections.

If all the saddles have negative saddle quantity, liftoff cannot occur at any fixed point and, upon the addition of small noise, the distribution of solutions traveling around the network will form "tubes" centered on the cycles; the widths of the tubes will be proportional to the rms of the noise. In this case a noise tube will intersect with the linear approximation to the unstable manifold at ξ_2 as suggested in Fig. 6(a). A noisy trajectory will switch randomly between visits to the three-cycle and the four-cycle (Fig. 3), with the proportion of visits made to each cycle depending on the relative size of the intersection of the noise tube and the local basins of attraction of each cycle (see Sec. III A).

If all saddles except ξ_1 have negative saddle quantity,

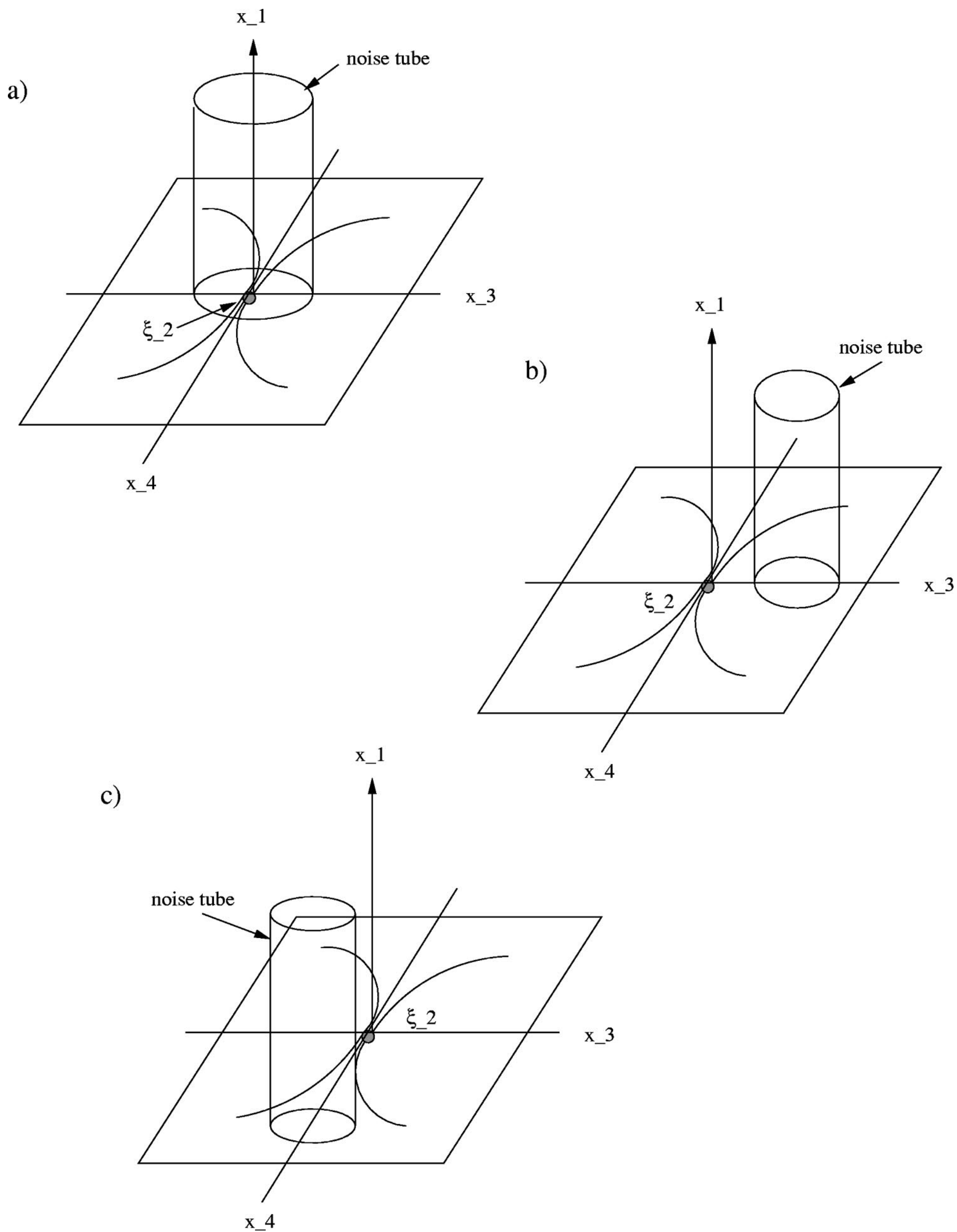

FIG. 6. Relative positions of the noise tube and the local basins of the two cycles. (a) No liftoff at ξ_1, so noise tube is centered on the heteroclinic connection. (b) Liftoff at ξ_1 in the three-direction only ($c_{13}<e_{12}<c_{14}$), noise tube shifted in the x_3 direction. (c) Liftoff at ξ_1 in the four-direction only ($c_{14}<e_{12}<c_{13}$), noise tube shifted in the x_4 direction.

there are two cases, depending on the eigenvalues of ξ_1. If there is liftoff at ξ_1 in the three-direction only ($c_{13}<e_{12}<c_{14}$), the center of the noise tube connecting ξ_1 to ξ_2 will be shifted away from the determinisitic connection and in the x_3 direction. See Fig. 6(b). Thus, for small enough noise and strong enough liftoff, the noise tube will fall squarely into the basin of the three-cycle. We see that with the inclusion of noise all initial conditions will result in trajectories that eventually end up traversing the three-cycle with little chance of switching to the four-cycle (see Fig. 4).

On the other hand, if there is liftoff at ξ_1 in the four-direction only ($c_{14}<e_{12}<c_{13}$), the center of the noise tube connecting ξ_1 to ξ_2 will be shifted in the x_4 direction as shown in Fig. 6(c). For small enough noise and strong enough liftoff the four-cycle will thus dominate the noisy dynamics, regardless of which cycle is seen in the deterministic dynamics.

The ''blocking'' of cycles, illustrated in Fig. 5, is a more subtle case, and is seen in the case of weak liftoff at ξ_1 in the direction of x_4. In this case, the noise tube intersects the local basins of both cycles, resulting in intermittent switching between the cycles. See Sec. III B for details.

Next we take up each case in turn, documenting what predictions can be made about the switching frequency and why.

A. The memoryless cycle

In the case where the saddle at ξ_1 has saddle quantity significantly less than 0, we say the cycle is ''memoryless'' because the contraction of the flow at ξ_1 combined with the noise removes any liftoff effects or deterministic biases due to the passage near the other saddle points in the cycle. Then the stochastic dynamics can be analyzed just in the neighborhood of the two-dimensional unstable manifold at the ξ_2 saddle point, and a scaling law for the proportion of orbits that traverse the four-cycle as noise is increased can be com-

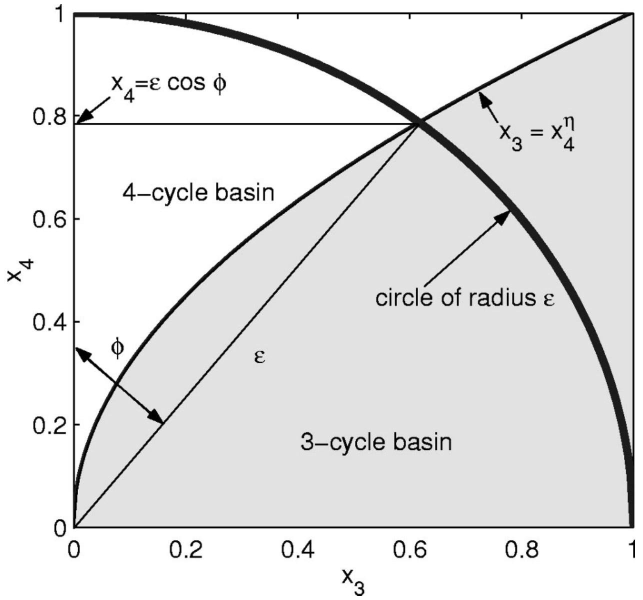

FIG. 7. Basins of attraction for the three-cycle and four-cycle with a noise tube (radius ϵ) centered at the origin.

puted. Our claim that the cycle is memoryless in this case is encapsulated in the first assumption below; the success of the scaling law we obtain in part justifies this assumption and the one following, which are both used in deriving the law.

Assumption 1: If the saddle quantity at ξ_1 is sufficiently negative, then the probability distribution of orbits leaving a box-neighborhood of ξ_1 in the direction of ξ_2 is well approximated by a Gaussian centered on the heteroclinic connection between ξ_1 and ξ_2.[6]

Assumption 2: Upon entrance to a box-neighborhood of the ξ_2 saddle point, the solutions then have a probability distribution that can be approximated by a Gaussian in the x_3 and x_4 coordinates and that is centered on the heteroclinic connection from ξ_1 to ξ_2.

The number of trajectories going to the four-cycle is determined by the overlap between the incoming probability distribution at ξ_2 and the cusp-shaped region through which trajectories pass on their way around the four-cycle. In particular, if both of the assumptions hold, the number of trajectories going to the four-cycle can be found by integrating a Gaussian distribution centered at the origin of the x_3,x_4 plane over the cuspoidal region $x_3 < c x_4^\eta$ where x_3 and x_4 are positive, $\eta = e_{23}/e_{24}$, and c is a positive constant. Note that by assumption $\eta > 1$.

A first order approximation to this calculation is given by integrating a flat probability distribution which drops to zero at a distance ϵ from the origin over the same domain. ϵ is taken to be proportional to the rms noise level. This reduces to computing the area of intersection of a disc of radius ϵ and the cuspoidal region defined above (see Fig. 7).

Since we cannot know the exact position of the separatrix between orbits that go towards ξ_3 and orbits that go towards ξ_4 without knowledge of the global dynamics (i.e., we cannot find c), we further restrict the calculation to finding the *scaling* of that area as ϵ increases. This also allows us to make a further approximation: the area of intersection of the disc and the cuspoidal region is underestimated by the area under the separatrix bounded by $x = \epsilon \cos \phi$, where ϕ is the angle determined by the intersection of the circle of radius ϵ and the separatrix. This area is proportional to $\epsilon^{\eta+1}$, and an overestimate of the cusp with the same scaling properties can be easily found. The ratio of that amount to the total area under the flat distribution is $\epsilon^{\eta-1}$. Hence the proportion of trajectories traversing the four-cycle should scale like $\epsilon^{\eta-1}$ with respect to the rms noise level, ϵ.

To confirm this scaling result we performed numerical simulations of the network with additive white noise on each coordinate, under various conditions. Each test has $e_{24} = 2.0$ and $e_{23} = 2.5$, and ξ_1 has significantly negative saddle quantity. The rest of the eigenvalues were set to explore different possibilities for the global dynamics as discussed in Ref. 7. The main features of the deterministic dynamics for each case, with the corresponding coefficient sets, are as follows.

(i) Test 1: The three-cycle is eas, and almost all trajectories started anywhere in the network are eventually attracted to it. $(c_{13},c_{14},c_{21},c_{32},c_{34},c_{42},c_{43},e_{12},e_{31},e_{41}) = (2.2,6.2,4.3,1.9,3.8,3.7,4.8,1.5,2.0,4.8)$.

(ii) Test 2: The four-cycle is not eas, but almost all trajectories started anywhere in the network are eventually attracted to it. $(c_{13},c_{14},c_{21},c_{32},c_{34},c_{42},c_{43},e_{12},e_{31},e_{41}) = (4.8,3.2,1.3,10.0,2.0,10.0,2.8,1.8,1.0,2.5)$.

(iii) Test 3: The three-cycle is eas, but the four-cycle attracts an open set of initial conditions. $(c_{13},c_{14},c_{21},c_{32},c_{34},c_{42},c_{43},e_{12},e_{31},e_{41}) = (2.2,4.2,4.3,2.4,3.0,3.7,7.0,1.8,2.0,2.8)$.

(iv) Test 4: Neither of the cycles is eas, but both attract open sets of initial conditions with the four-cycle attracting a larger set of initial conditions than in the previous case. $(c_{13},c_{14},c_{21},c_{32},c_{34},c_{42},c_{43},e_{12},e_{31},e_{41}) = (6.2,4.2,7.3,2.4,12.7,5.7,5.0,0.5,2.0,4.8)$.

For a given coefficient set integrations were performed with increasing noise levels, where for each noise level a 1000 time unit transient was run out (between 50 and 100 cycles depending on the noise level), and the orbit was then allowed to evolve until it had made 10 000 passages past ξ_2. (Note that the memoryless nature of the dynamics means that removing the transient and then following the evolution of a single trajectory is equivalent to sampling from an ensemble of initial conditions.) The number of traversals of the four-cycle was counted and that percentage recorded for each noise level. A log-log plot of the data for four tests is shown in Fig. 8. A linear regression of each test yields slopes that are within 8% of each other, and within 6% of the theoretical value $2.5/2.0 - 1 = 0.25$, specifically: 0.257, 0.236, 0.240, and 0.251 for tests 1, 2, 3, and 4 respectively.

From the evidence of these experiments it appears that the global dynamics of the network is not important in determining switching rates once noise is added to the system, so long as ξ_1 has saddle quantity significantly less than 0. The proportion of orbits that traverse the four-cycle is determined by the dynamics local to ξ_2 and specifically by the ratio of eigenvalues e_{23}/e_{24}, with the noise simply allowing the system to sample randomly from the two regions, i.e., the local region of orbits that deterministically go towards ξ_3

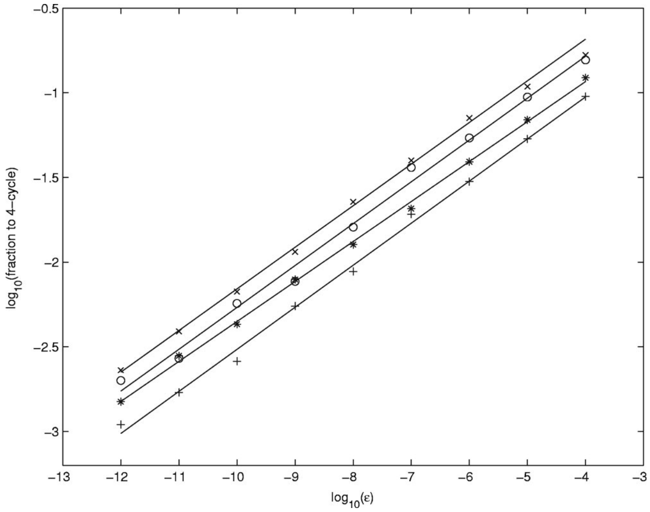

FIG. 8. Scaling of switching percentages with noise level. For details on the four tests plotted see the text, with the designations: test 1 (+), test 2 (×), test 3 (○) test 4 (*). In each case ξ_1 has negative saddle quantity and variations are in the attractivity properties of the other fixed points.

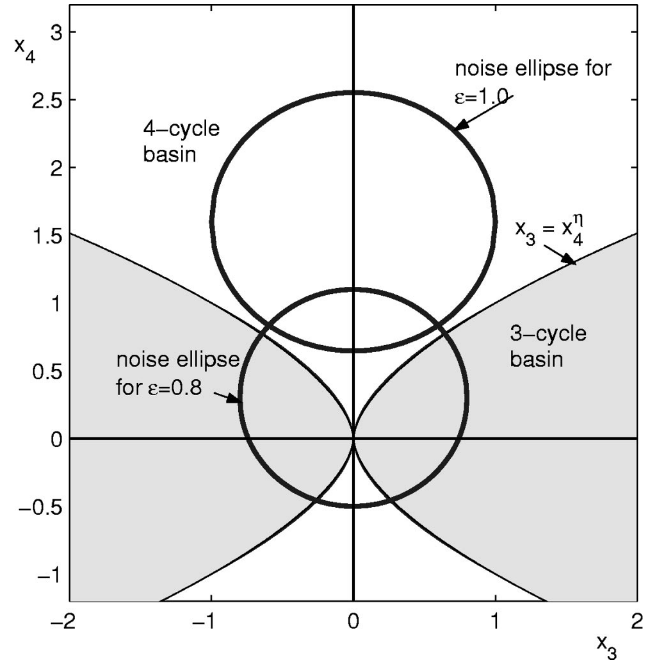

FIG. 9. Basins of attraction for the three- and four-cycles with noise ellipses lifted-off in the x_4 direction, two noise levels.

and the local region of orbits that go towards ξ_4. This should be contrasted with the cases outlined in the next section, where the noise interacts with the deterministic dynamics to produce subtle and sometimes counterintuitive results.

B. Interaction of noise and dynamics

If ξ_1 has positive saddle quantity, then the details of the deterministic dynamics determine how the noise will influence switching of orbits from one cycle to the other. The critical part of the network for this is the passage near the fixed point ξ_1, with the liftoff occurring at this saddle being the crucial feature of the dynamics.

In the following, we develop our argument by considering first the case where there is liftoff at ξ_1 in the x_4 direction but not in the x_3 direction. We assume that ξ_3 and ξ_4 both have negative saddle quantity, so that the probability distribution of orbits entering a neighborhood of ξ_1 from the direction of either ξ_3 or ξ_4 is Gaussian, being centered on the appropriate heteroclinic connection and with variance of order ϵ (see Assumptions 1 and 2 and Ref. 6). In this case, in the limit of small ϵ, the outgoing probability distribution at ξ_1 is Gaussian, with mean shifted from the heteroclinic connection in the x_4 direction by a distance proportional to $\epsilon^{c_{14}/e_{12}}$, and with variances in the x_3 and x_4 directions being proportional to ϵ and $\epsilon^{c_{14}/e_{12}}$, respectively. In the x_4 direction, the variance is smaller than the mean in the limit $\epsilon \to 0$, i.e., we have liftoff in the x_4 direction.[6]

Now, as discussed in Sec. I, noise has a negligible effect on the dynamics along the heteroclinic connection, away from the fixed points. Furthermore, as discussed in Ref. 7, the symmetries of system (1) are such that the deterministic dynamics does not "twist" the distribution as it is transported along the heteroclinic connection from ξ_1 to ξ_2, but merely multiplies the x_3 and x_4 coordinates by constant factors. Hence, the distribution of trajectories arriving from the direction of ξ_1 as they pass through the side of a box-neighborhood of ξ_2 will again be an off-center Gaussian with dimensions that scale as the Gaussian above in the limit of small noise. We refer to this distribution (i.e., the intersection of the noise tube and the side of the box-neighborhood) as the "noise ellipse."

As mentioned earlier, the region where orbits going towards ξ_4 intersect the incoming side of a box-neighborhood of ξ_2 is cusp-shaped, and corresponds to the region $x_3 < c x_4^\eta$ $(x_3, x_4 > 0)$, and its reflection about the x_3 axis, in Fig. 9. The position of the ellipsoidal distribution of trajectories relative to this cusp will determine the qualitative dynamics of the noisy network, and this position varies with the size of ϵ. In determining the relative positions of the noise ellipse and the cusp-shaped region, we make a first-order approximation to the distribution as in the previous subsection, i.e., we replace the true distribution, which decays smoothly to zero as the distance from the center is increased, with a flat distribution that is nonzero on an ellipse with center at the mean of the true distribution, with semi-major axes parallel to the x_3 and x_4 axes and of length equal to the variances in those directions. This approximation yields scaling arguments about the behavior of the trajectories as noise level (rms) is varied, just as in the previous section.

Whether or not the ellipse lies completely inside the cusp is approximately determined by the width (in the x_3 coordinate) of the cusp at the x_4 value corresponding to the center of the ellipse, i.e., at x_4 proportional to $\epsilon^{c_{14}/e_{12}}$. The width of the cusp at this point is proportional to ϵ^α where $\alpha = e_{23} c_{14} / e_{24} e_{12}$ and the width of the ellipse in the x_3 direction is proportional to ϵ. If $\alpha < 1$, then as $\epsilon \to 0$ the cusp becomes wider than the noise ellipse and hence trajectories will with high probability follow the four-cycle. If $\alpha > 1$, then as $\epsilon \to 0$ the cusp closes faster than the noise ellipse, and a greater fraction of the orbits will follow the three-cycle.

A similar argument holds when there is liftoff at ξ_1 in the x_3 direction only, with ξ_3 and ξ_4 having sufficiently negative saddle quantities. However, in this case we find that the exponent analogous to α is $\beta = e_{24}c_{13}/e_{23}e_{12}$ and note that $\beta < 1$ whenever there is liftoff in the x_3 direction since $e_{24} < e_{23}$ by assumption. Thus liftoff in the x_3 direction always makes traversals of the four-cycle extremely unlikely for small noise.

Depending on the details of the liftoff near the fixed point ξ_1 and on the deterministic dynamics of the network as a whole, we find three qualitatively different cases:

(1) There is liftoff at ξ_1 in the x_3 direction. This can occur when the three-cycle deterministically attracts almost all trajectories (see Fig. 4), in which case small noise has little effect on the switching behavior of trajectories. However, it can also occur in cases where the three-cycle is eas but the four-cycle attracts open sets of initial conditions; in this case, the dynamics is modified by the noise so that the three-cycle occurs predominantly for all initial conditions. The system with the coefficient set $(c_{13}, c_{14}, c_{21}, c_{32}, c_{34}, c_{42}, c_{43}, e_{12}, e_{23}, e_{24}, e_{31}, e_{41}) = (0.5, 3.3, 4.3, 4.9, 3.8, 3.7, 4.8, 3.5, 2.5, 2.0, 1.0, 4.8)$ is an example of this situation.

(2) There is strong liftoff at ξ_1 in the x_4 direction (α significantly smaller than 1), making traversals near the three-cycle very unlikely for small noise. This can occur when the four-cycle deterministically attracts almost all trajectories (in which case small noise does not affect the switching behavior) or when the three-cycle is eas and the four-cycle attracts an open set of trajectories (in which case two cycles are seen in the deterministic dynamics, depending upon the initial conditions, but independent of initial condition the noisy case favors the four-cycle). An example of the latter case is discussed further below.

(3) The liftoff in the x_4 direction is weak enough that there is a significant probability of traversals near the three-cycle (α larger than in case 2 and close enough to one). In this case we will see orbits switching between traversals around the three-cycle and around the four-cycle. This can occur when one cycle attracts almost all initial conditions deterministically or when both cycles attract large sets of initial conditions deterministically. In the latter situation, we encounter intermittent behavior in the noisy system: a trajectory loops many times around the same cycle before it switches to the other cycle. See Fig. 5.

We describe more fully below some situations that generate these cases.

Figure 4 is an illustration of a case that is left qualitatively unaltered by the addition of small noise. In this case the three-cycle deterministically attracts almost all initial conditions. The liftoff is strongly into the x_3 direction and hence the probability of switching to a four-cycle is very small.

The behavior for α significantly less than one is illustrated by a simulation with the following parameters: $(c_{13}, c_{14}, c_{21}, c_{32}, c_{34}, c_{42}, c_{43}, e_{12}, e_{23}, e_{24}, e_{31}, e_{41}) = (6.2, 1.5, 7.3, 2.4, 12.7, 5.7, 5.0, 3.8, 2.5, 2.0, 2.0, 4.8)$. Note that this gives $\alpha = 0.49$. With these parameters, the three-cycle is eas in the deterministic system, so once a trajectory falls into the basin of attraction of the three-cycle it will not leave it in the absence of noise. Similar behavior is observed in simulations with noise of rms 10^{-12}; trajectories starting on the three-cycle follow this cycle for $t = 2 \times 10^4$, although a switch is subsequently made to the four-cycle and the four-cycle then persists for a long time.

However, if we increase the noise, the liftoff in the x_4 direction becomes large enough to switch trajectories immediately to the four-cycle, and since $\alpha < 1$ the probability of switching back is low. For example, for $\epsilon = 10^{-5}$ an orbit started on the three-cycle will visit only the four-cycle for $t < 10^4$. However, the probability of switching to the three-cycle is not zero, and longer simulations can uncover occasional single or double traversals of the three-cycle. An exact analysis of the probability of switching in these situations would require a detailed description of the deterministic dynamics away from the saddle points in the cycle in each case, and could not be generalized.

Intermittent behavior is illustrated in Fig. 5. The parameters are such that deterministically both cycles attract significant sets of initial conditions, and the three-cycle is eas. In this case we have liftoff in the x_4 direction but $\alpha = 0.83$ is close enough to one to give a small but finite probability of an orbit landing in the basin of attraction of the three-cycle. Once there, the deterministic dynamics tends to keep the orbit away from all the basin boundaries except for the boundary abutting the three-cycle; as long as the trajectory does not pass too close to the four-cycle the noise will not cause it to leave the basin of attraction of the three-cycle. Eventually, however, the orbit will get sufficiently close to the heteroclinic cycle that the liftoff in the x_4 direction will cause it to fall into the basin of attraction of the four-cycle. The orbit will then stay near the four-cycle for some time since the four-cycle also attracts a large set of initial conditions, and the noise-induced probability to leave the basin of attraction is small. Note that while the average length of the block increases with decreasing noise, we find switching blocks even at a noise level of 10^{-12}.

We illustrate the counterintuitive action of the noise with one final example. For $(c_{13}, c_{14}, c_{21}, c_{32}, c_{34}, c_{42}, c_{43}, e_{12}, e_{23}, e_{24}, e_{31}, e_{41}) = (2.6, 1, 4.3, 1, 2.0, 10.0, 2.8, 1.8, 2.5, 2.0, 1.2, 1.5)$ the three-cycle is eas. Hence for initial conditions in the basin of attraction of the three-cycle, and for very small noise (10^{-12}), we find only passages near the three-cycle for $t \leq 10^4$. However, there is liftoff in the x_4 direction with $\alpha = 0.83$. As a result, for $\epsilon = 10^{-8}$, a trajectory that starts on the three-cycle switches over to the four-cycle and stays there for $t \leq 10^4$. For $\epsilon = 10^{-5}$ the probability of switching to the three-cycle has increased again (as a consequence of the noise ellipse getting larger as ϵ increases) and we find blocking behavior for $t \leq 10^4$. In fact, long-time simulations at each noise level reveal blocking in all these cases, with substantially more traversals of the four-cycle per block for $\epsilon = 10^{-8}$ than for $\epsilon = 10^{-5}$, which could lead to the conclusion that one cycle is favored over the other for all time at that noise level. At $\epsilon = 10^{-12}$ a switch to the four-cycle is seen after about $t=5\times10^4$, and it persists for an equally long period of time.

In this section we have discussed situations where there is liftoff only in one direction. It is possible to have liftoff in both the x_4 and x_3 directions simultaneously without violating the overall stability properties of the cycles. In this situation the noise ellipse will be centered at a point with neither x_3 or x_4 equal to zero, corresponding to a point in the upper right quadrant in Fig. 9. Most generally this will lead to switching as in Sec. III A, but the statistics will not scale as calculated there. Rather the switching percentage will depend on both the size and position of the origin of the ellipse, and on the overall attractivity properties of each cycle, in a way that would have to be calculated on a case-by-case basis. The latter is true also in the case that the equilibrium ξ_1 has negative saddle quantity and also either ξ_3 or ξ_4 or both have positive saddle quantity.

IV. CONCLUSION

Systems containing heteroclinic networks may have limit sets with very complicated attractivity properties and which are neither asymptotically stable nor essentially asymptotically stable. For a purely deterministic system this can lead to a highly complicated taxonomy of different cases describing the long term behavior of trajectories (as in, for example Ref. 7), and it might be thought that making predictions about the effect of noise in these cases would be intractable. However, by considering the effect of noise on a simple heteroclinic network of the type studied by Ref. 7, we have found general mechanisms that explain the interplay between dynamics and noise and which allow us to make straightforward predictions about the behavior of general heteroclinic networks with noise, including but not restricted to those cases which have complicated attractivity properties.

Specifically, our results should apply to any heteroclinic network in which two or more heteroclinic cycles share a one dimensional heteroclinic connection from an equilibrium ξ_{n-1} to an equilibrium ξ_n and where ξ_n has an unstable manifold of dimension higher than one. In particular, if ξ_{n-1} has negative saddle quantity, then under addition of noise, trajectories in the system will randomly visit all the cycles that originate at ξ_n, with the percentage of times each cycle is traversed being determined by the linearized deterministic dynamics near ξ_2.

On the other hand, if ξ_{n-1} has positive saddle quantity and there is sufficiently strong noise-induced liftoff in a particular direction at ξ_{n-1}, this will typically force the occurrence of traversals primarily near one cycle, and the behavior of an individual trajectory will not show any stochastic behavior on a scale larger than the applied noise. This noise induced behavior has a persistence property: as long as the global stability of the underlying cycles is maintained and as long as the liftoff property persists, the exact values of the other parameters in the system will not determine the switching behavior.

In our example system (1), we have seen that there exists a third type of behavior characterized by intermittent switching between different cycles. This behavior occurs when both cycles in the deterministic network attract significant sets of initial conditions. The frequency of the switching will be strongly determined by the details of the global attractivity properties of the respective heteroclinic cycles; for instance, the cases in Ref. 7 would have to be studied individually to determine the effect of even very small additive noise. We note that this intermittent switching is unlikely to occur in heteroclinic networks such as those in Ref. 14 where different heteroclinic cycles with a common heteroclinic connection cannot simultaneously attract open sets of trajectories from a neighborhood of the network.

We have described here a variety of responses of the network to very small noise and have demonstrated a rich array of phenomena that can be observed. It was originally thought that small additive noise would wipe out the unique features of the network, and the "strong" three-cycle would predominate. This is clearly not the case, and in fact the noise can *reinforce* the the "weak" four-cycle so that it occurs *more often* than it would without noise. Perhaps most interesting is the interplay of the noise and the dynamics as described in Sec. III B. Unraveling the details of such interactions in other systems, such as networks with complex eigenvalues, would be an obvious continuation of this work.

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