AVERAGED MODELS FOR LONG SUPPLY CHAINS WITH RANDOM BREAKDOWNS

OUTLINE

- Previous work: Deterministic models for long supply chains, leading to hyperbolic conservation laws (dieter, pierre et moi $\rightarrow$ Herty+Klar for networks)

- Introduce stochasticity into the model. (Random breakdown of processors)

- An argument against queueing theory: non-Markov processes

- Random particle formulation of the conservation laws in a Lagrangian frame

- Mean field theory $\Rightarrow$ equation for expected densities.

The Automaton

Goal: Define a simple (deterministic) model for one processor with one buffer queue in front of it. Take a continuum limit for a long chain of processors. $\Rightarrow$ Obtain a PDE for the client density.

Principles

- The processor has a processing time T and a processing rate μ (i.e. can handle μT clients at the same time).

- Client Nr. n arrives in a buffer queue at arrival times a_n, is fed into the processor at time b_n, leaves the processor (and enters the next queue) at a time $e_n = b_n + T_n$.

A simple rule for a_n and b_n

Case 1: (full queue)

$b_n = b_{n-1} + \frac{1}{\mu}$ if $b_{n-1} + \frac{1}{\mu} > a_n$ (if the part has arrived).

Case 2: (empty queue)

$b_n = a_n$ if $b_{n-1} + \frac{1}{\mu} < a_n$ (if the part has not yet arrived).

recursion: $b_n = \max\{b_{n-1} + \frac{1}{\mu}, a_n\}$

$$e_n = b_n + T : \qquad e_n = \max\{e_{n-1} + \frac{1}{\mu}, a_n + T\}$$

LARGE QUEUEING SYSTEMS - CONTINUUM LIMITS

Consider a large number of queues and processors $m = 1, .., M >> 1$.
Let $a_n^m = \tau_n^m$, $e_n = \tau_n^{m+1}$, $T_m = \frac{1}{V_m}$.

$$\tau_n^{m+1} = \max\{\tau_{n-1}^{m+1} + \frac{1}{\mu_m}, \tau_n^m + \frac{1}{V_m}\}$$

Continuum limits:

- Introduce a stage variable $m \to x \in [0,1]$.
- Parts enter at stage $x = 0$ and exit at stage $x = 1$.
- One buffer + one processor correspond to an interval of length Δx.
- Continuous part index $n \to y$.
- $\tau_n^m \to \tau(x, y)$

$$0 = \max\{-\partial_y \tau + \frac{1}{\mu(x)}, -\partial_x \tau + \frac{1}{V(x)}\}$$

Conservation laws in Eulerian and Lagrangian formulation

$\tau(x, y)$ is monotone in x and y (but not strictly monotone in y!) $\Rightarrow$

$$\boxed{t = \tau(x, y) \iff y = u(x, t) \iff x = \xi(y, t)}$$

- $t = \tau(x, y)$: time part nr. y arrives at processor x
- $y = u(x, t)$: number of the part at processor x at time t (N-curve).
- $x = \xi(y, t)$: position (processor number) of part nr. y at time t.
- Flat regions of τ correspond to jumps in u.

Formulations of the conservation law

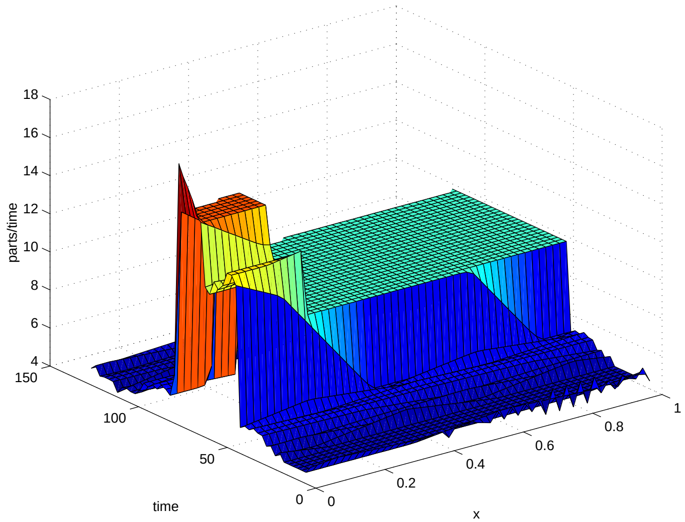
$0 = \max\{-\partial_y \tau + \frac{1}{\mu(x)}, -\partial_x \tau + \frac{1}{V(x)}\}$
$\partial_t u(x,t) = \min\{\mu, -V\partial_x u\}$
$\partial_t \xi(y,t) = \min\{-\mu\partial_y \xi, V\}$

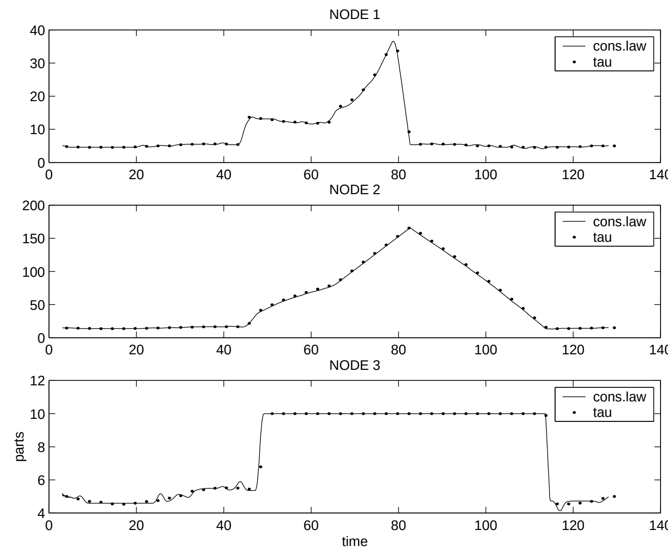
Variables:

- $\rho = -\partial_x u$: density, $\phi = \partial_t u$: Flux
- $\partial_t \xi$: velocity, $\partial_y \xi$: specific volume.

$$\boxed{\partial_t \rho + \partial_x \phi = 0, \quad \phi = \min\{\mu, V\rho\}}$$

u and ϕ can become discontinuous $\Rightarrow$ the density $\rho(x,t)$ can exhibit $\delta-$ function concentrations.
This corresponds to bottlenecks and transient build - ups in the buffers.





3 processors, discontinuous N-curve u, WIP=$\Delta x\rho$

Steady states

$$\boxed{\partial_t \rho + \partial_x \phi = 0, \quad \phi = \min\{\mu, V\rho\}}$$

- Influx condition: $\phi(0,t) = \lambda(t)$ parts / time arriving in the chain.
- Steady state: $\partial_t \rho = 0$, $\phi = const$

Two cases:

- Case 1: equilibrium $\lambda \leq \min_x\{\mu(x)\} \Rightarrow \phi = \lambda = V\rho$

- Case 2: bottleneck $\lambda > \min_x\{\mu(x)\} \Rightarrow$ growth of queues ($\delta-$ functions)

Queueing Theory

Goal: Make the system stochastic.

- Standard textbook queueing theory usually provides steady state results in a stochastic setting.
- 'Clients' arrive in random intervals Δa with $d\mathcal{P}[\Delta a = \frac{1}{s}] = \mathcal{A}(s)ds$ with $\mathcal{A}$ the distribution of the arrival rate
- Each time a client reaches the front of the queue 'it' is served and we pick a random time $\Delta\tau$ when we serve the next client according to $d\mathcal{P}[\Delta\tau = \frac{1}{s}] = \mathcal{S}(s)ds$ with $\mathcal{S}$ the distribution of the service-rate
- In steady state ($\mathcal{A}$, $\mathcal{S}$ time independent) there are tons of formulas for expectation and variance of the resulting queue length q.

Example: The $M/M/1$ queue. ($\mathcal{A}$ and $\mathcal{S}$ are exponential distributions)

$$\langle q \rangle = \frac{\langle \mathcal{A} \rangle}{\langle \mathcal{S} \rangle - \langle \mathcal{A} \rangle} (= \frac{\lambda}{\mu - \lambda})$$

Compare to $\lambda = \min\{\mu, V\rho\} \;\Rightarrow\; \rho = \frac{\lambda}{V}$ if $\lambda < \mu$

Stochastic Aspects of the System

Processor breakdown:

Distribution of time to failure $d\mathcal{P}[\tau_{up} = s] = \mathcal{U}(s)ds$

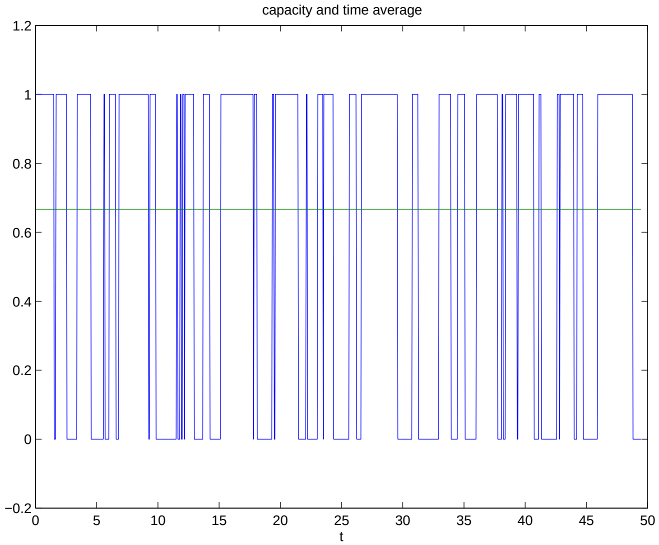
Distribution of time to 'repair' $d\mathcal{P}[\tau_{down} = s] = \mathcal{D}(s)ds$

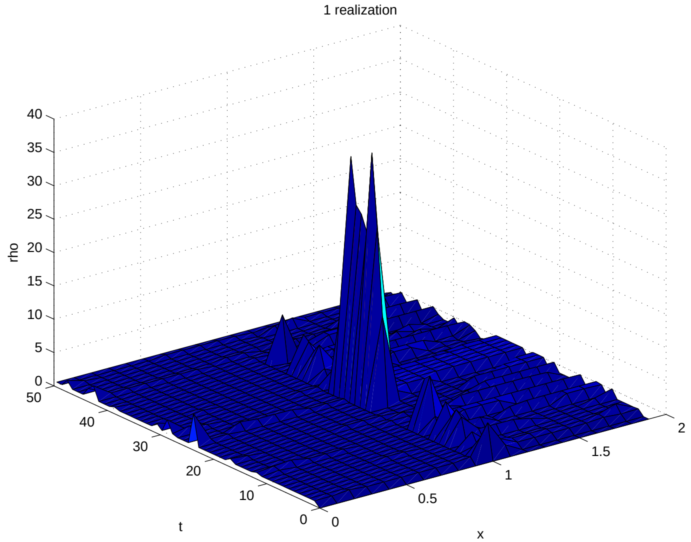
Incorporate this into the fluid dynamic model by setting the capacity μ to zero during breakdowns

$$\partial_t \rho + \partial_x \min\{\mu(x,t), V(x)\rho\} = 0$$

$\mu(x,t)$ is a random time dependent variable, assuming the values of $\mu = c(x)$ (when the processor is 'on') and $\mu = 0$ (when it is 'off').

- Processor has just turned on at time t_0
- Pick Δt_{up} according to distribution $\mathcal{U}$ and Δt_{down} according to distribution $\mathcal{D}$.
- Set $\mu(t) = c$ for $t_0 < t < t_0 + \Delta t_{up}$
and $\mu(t) = 0$ for $t_0 + \Delta t_{up} < t < t_0 + \Delta t_{up} + \Delta t_{down}$





One realization of μ and corresponding realization of ρ.

The difference to queuing theory

- In QT we pick a random time when the next client is served. We know all parameters in the distribution from the state of the system at this time.
- In our model the time to the next service is either $\frac{1}{\mu}$ (which is essentially deterministic), if the processor remains 'on', or the next time the processor turns 'on' again, when the processor is 'off'.
- In either case, the time to next service depends on the time the processor has been 'on' or 'off', i.e. on the history of the system, and cannot be determined by the current state alone.
- This results into a non - Markovian stochastic process, inherently different from the model given in standard QT.

Stochastic Version of the Fluid Model

- Start from the Lagrangian formulation
- Discretize by a random particle method
- Derive a high dimensional Boltzmann transport equation for the particles and processor states
- Molecular chaos - mean field theory
- Long time averages

Lagrangian model

$$\partial_t \xi(y,t) = v(y,\mu,\xi) = \min\{\frac{\mu(\xi,t)}{\rho(\xi,t)}, V(\xi)\}$$

$\partial_t \xi$: velocity of part y, $\rho(\xi,t) = -\frac{1}{\partial_y \xi}$: density

$\boxed{\rho(x,t) = \int \delta(x-\xi(y,t))\ dy}$

Particle discretization:
Discretize y: $\xi_n(t),\ n = 1,..,N$

$$\partial_t \xi_n(t) = \min\{\frac{\mu(\xi_n,t)}{\rho(\xi_n,t)}, V(\xi_n)\}$$

$$\boxed{\rho(x,t) = \Delta y \sum_{n=1}^{N} \delta(x - \xi_n(t))}$$

In the particle formulation ρ is a measure! $\frac{1}{\rho}$ will have to be approximated.

$\frac{1}{\rho}$: specific volume (spacing of particles)

$$\boxed{\Rightarrow \ \frac{1}{\rho(\xi_n,t)} = \min\{\frac{\xi_k-\xi_n}{\Delta y} : \ \xi_k > \xi_n\}}$$

$$\partial_t \xi_n(t) = v_n(\overrightarrow{\xi}, \overrightarrow{\mu}) = \min\{V(\xi_n), \ \mu(\xi_n,t)\frac{\xi_k - \xi_n}{\Delta y} : \ \xi_k > \xi_n\}$$

The generation of $\mu(x,t)$

Change randomly between $\mu = c$ and $\mu = 0$ with a scattering frequency ω

$$\mathcal{P}[r=1] = \Delta t\omega(x,\mu(x,t)),\ \mathcal{P}[r=0] = 1 - \Delta t\omega$$

$$\mu(x,t+\Delta t) = [1-r]\mu(x,t) + r[c(x) - \mu(x,t)]$$

This yields exponential distributions $\mathcal{U},\mathcal{D}$ for the up and down times τ_{up} and τ_{down}

$$\frac{d\mathcal{P}}{ds}[\tau_{up}(x) = s] = \mathcal{U}(x,s) = \omega(x,c(x))e^{-\omega s}$$

$$\frac{d\mathcal{P}}{ds}[\tau_{down}(x) = s] = \mathcal{D}(x,s) = \omega(x,0)e^{-\omega s}$$

One time step of one realization of the random particle method

$$\boxed{\mu(x,t+\Delta t) = [1-r]\mu(x,t) + r[c(x) - \mu(x,t)]}$$

$$\boxed{\xi_n(t+\Delta t) = \xi_n(t) + \Delta t v_n(\xi,\mu),}$$

$v_n(\overrightarrow{\xi}, \overrightarrow{\mu}) = \min\{V(\xi_n),\ \mu(\xi_n,t)\frac{\xi_k-\xi_n}{\Delta y} :\ \xi_k > \xi_n\}$
$\mathcal{P}[r=1] = \Delta t\omega(x,\mu(x,t)),\ \mathcal{P}[r=0] = 1-\Delta t\omega$

Goal: Compute the expectation of ρ

$$\rho(x,t) = \Delta y \sum_{n=1}^{N} \delta(x - \xi_n(t))$$

Let $F(\overrightarrow{\xi}, \overrightarrow{\mu}, t)$ be the probability density for the whole particle ensemble and all the capacities at a given time t.

$$\langle \rho(x,t) \rangle = \Delta y \sum_{n=1}^{N} \int \delta(x-\xi_n) F(\overrightarrow{\xi}, \overrightarrow{\mu}, t)\ d\overrightarrow{\xi}\, d\overrightarrow{\mu}$$

For a given realization $\rho(x,t)$ is a measure. However $\langle \rho(x,t) \rangle$ is a smooth function.

Derive a Boltzmann - type transport equation for $F(\overrightarrow{\xi}, \overrightarrow{\mu}, t)$. Use a molecular chaos assumption. Build large time averages.

Final result:

For $\omega = \frac{1}{\langle\tau_{up/down}\rangle} >> 1$ we get

$$\partial_t\langle\rho(x,t)\rangle + \partial_x\phi_E(x,\langle\rho\rangle) = 0$$

$$\phi_E(x,\langle\rho\rangle) = \tau_{up}^{av}(x)c(x)[1-\exp(-\frac{V\langle\rho\rangle}{c})], \quad \tau_{up}^{av} = \frac{\langle\tau_{up}\rangle}{\langle\tau_{up}\rangle + \langle\tau_{down}\rangle}$$

Interpretation: Compare to $\phi = \min\{\mu, V\rho\}$

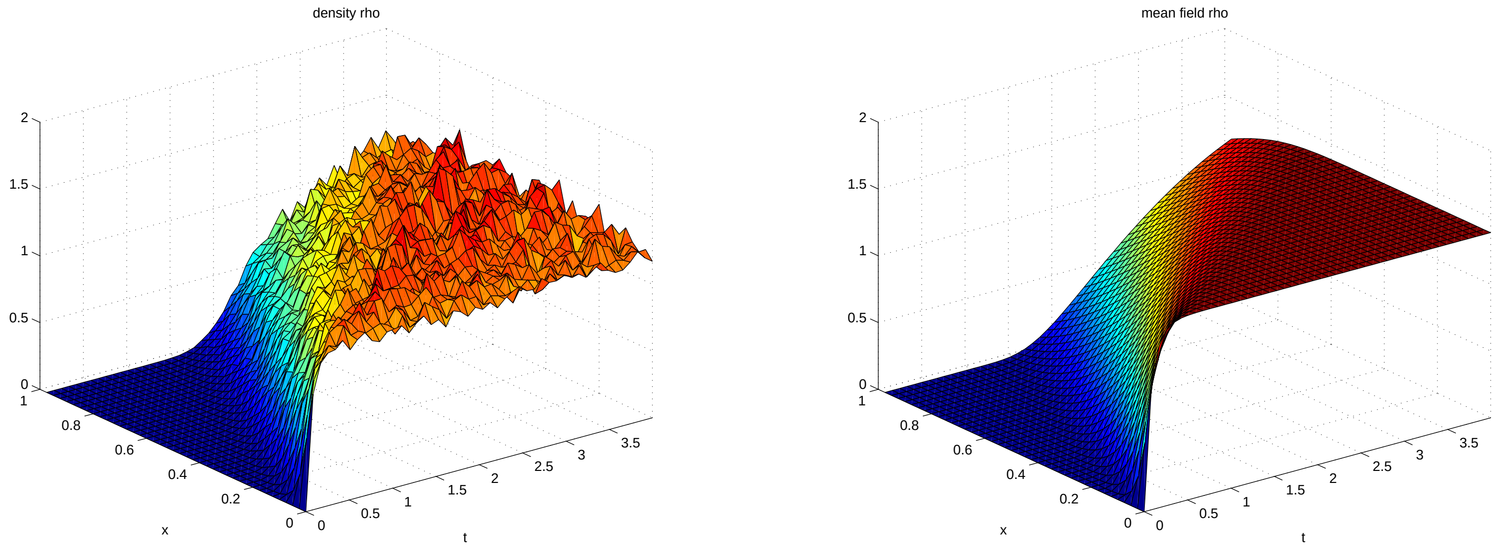
- μ is replaced by the 'on' capacity c.
- The function $\min\{c, V\rho\}$ is replaced by $c[1 - e^{-\frac{V\rho}{c}}]$, which has the same asymptotic properties for $\rho \to 0$ and $\rho \to \infty$.
- The whole flux ϕ is multiplied by the factor $\tau_{up}^{av} = \frac{\langle\tau_{up}\rangle}{\langle\tau_{up}\rangle+\langle\tau_{down}\rangle}$.

Experiments

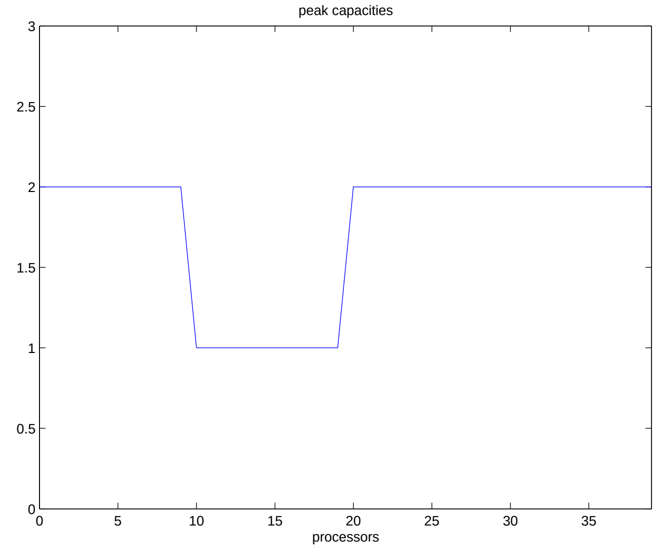
Experiment 1:

Quantitative comparison of steady states

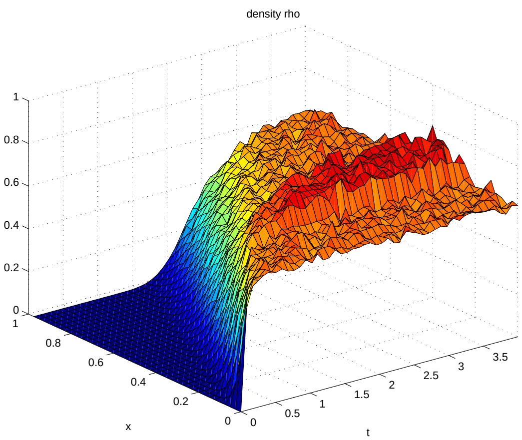
- 40 identical processors with a peak capacity $c = 2$ and $V = 1$.
- They are on half of the time $\langle \tau_{up} \rangle = \langle \tau_{down} \rangle = \frac{1}{20}$
- Solve with the flux $\phi = \min\{\mu, V\rho\}$ for 200 realizations of μ and average.
- Constant influx $\phi(x = 0) = 0.5$
- Compare to the solution for $\langle \rho \rangle$ of the equation or the mean.

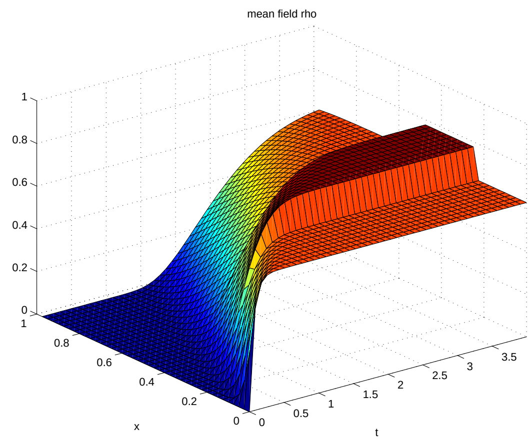


ρ averaged over 200 realizations and $\langle\rho\rangle$

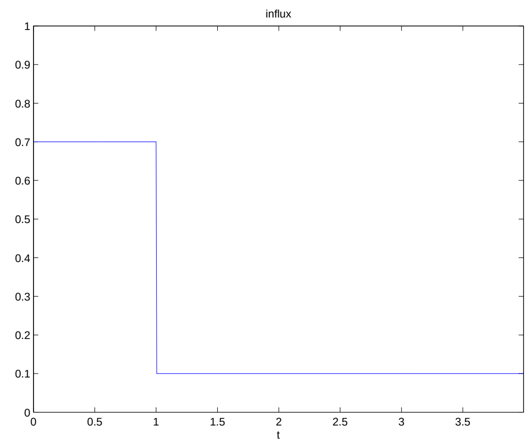


Experiment 2: A bottleneck in processors $11 - 20$

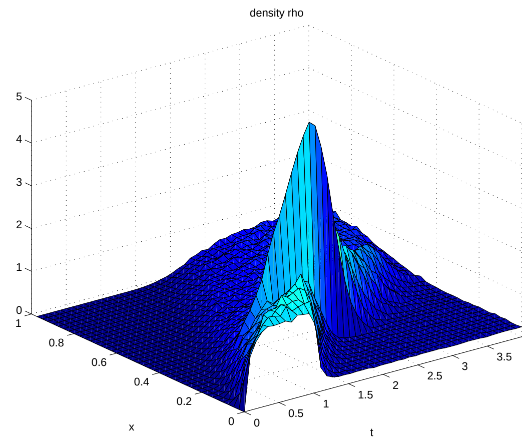




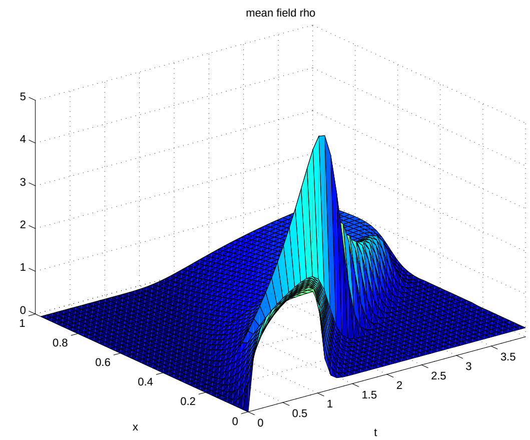
Comparison for constant influx $\phi(x = 0) = 0.25$



Experiment 3: The transient response, Influx



density rho
5
4
3
2
1
0
1
0.8
0.6
0.4
0.2
0
x
0
0.5
1
1.5
2
2.5
3
3.5
t



mean field rho
5
4
3
2
1
0
1
0.8
0.6
0.4
0.2
0
x
0
0.5
1
1.5
2
2.5
3
3.5
t

An overview over the derivation

The many-body probability density $F(\overrightarrow{\xi}, \overrightarrow{\mu}, t)$ with $\overrightarrow{\xi} = (\xi_1..\xi_N)$ and $\overrightarrow{\mu} = (\mu_1, .., \mu_M)$

$$\partial_t F(\overrightarrow{\xi}, \overrightarrow{\mu}, t) + \sum_n \partial_{\xi_n}[v_n(\overrightarrow{\xi}, \overrightarrow{\mu}, t) F] = \int Q(\overrightarrow{\mu}, \overrightarrow{\mu}') F(\overrightarrow{\xi}, \overrightarrow{\mu}', t)\, d\overrightarrow{\mu}'$$

$$Q(\overrightarrow{\mu}, \overrightarrow{\mu}') = \sum_m q_m(\mu_m, \mu'_m) \prod_{k \neq m} \delta(\mu'_k - \mu_k),$$

$$q_m(\mu_m, \mu'_m) = \omega_m(\mu'_m)[\delta(c_m - \mu'_m - \mu_m) - \delta(\mu'_m - \mu_m)] \ .$$

Molecular Chaos:

- Usually: $F(\overrightarrow{\xi}) = \prod_{n=1}^{N} f(\xi_n)$ particles are independent and identical.
- Independence for one realization of $\overrightarrow{\mu}$:
- $F^C(\overrightarrow{\xi}, \overrightarrow{\mu}, t)$: conditional probability of $\overrightarrow{\xi}$ for a given $\overrightarrow{\mu}$.

$$\boxed{F^C(\overrightarrow{\xi}, \overrightarrow{\mu}, t) = \frac{F(\overrightarrow{\xi}, \overrightarrow{\mu}, t)}{G(\overrightarrow{\mu}, t)}, \quad G(\overrightarrow{\mu}, t) = \int F(\overrightarrow{\xi}, \overrightarrow{\mu}, t)\, d\overrightarrow{\xi}}$$

$F^C \, d\overrightarrow{\xi}$ is a probability density for every $\overrightarrow{\mu}$!

Ansatz: $F^C(\overrightarrow{\xi}, \overrightarrow{\mu}, t) = \prod_{n=1}^{N} f^c(\xi_n, \overrightarrow{\mu}, t)$
Integrate out $\xi_2..\xi_N$ and compute the average velocity $u(\xi_1, \overrightarrow{\mu}, f^c)$ of the (now identical) particles in the limit $N \to \infty$, given by

$$u(\xi_1, \overrightarrow{\mu}, f^c) = \frac{\mu_m}{f^c}[1 - \mathrm{exp}(-\frac{V(\xi_1)f^c}{\mu_m})] \ .$$

for ξ_1 in the m-th processor

$$\partial_t[G(\overrightarrow{\mu}, t)f^c(\xi_1, \overrightarrow{\mu}, t)] + \partial_{\xi_1}[u(\xi_1, \overrightarrow{\mu}, f^c)Gf^c] =$$

$$\frac{1}{\varepsilon}\int Q(\overrightarrow{\mu}, \overrightarrow{\mu}')f^c(\xi_1, \overrightarrow{\mu}')G(\overrightarrow{\mu}') \ d\overrightarrow{\mu}',$$

where the probability density of the capacities $\overrightarrow{\mu}$ satisfies

$$\partial_t[G(\overrightarrow{\mu}, t)] = \frac{1}{\varepsilon}\int Q(\overrightarrow{\mu}, \overrightarrow{\mu}')G(\overrightarrow{\mu}', t) \ d\overrightarrow{\mu}',$$

Chapman Enskog for $\varepsilon << 1$.

Reduction to the classical case

$$\phi_E(x, \langle\rho\rangle) = \tau_{up}^{av}(x)c(x)[1-\text{exp}(-\frac{V\langle\rho\rangle}{c})], \quad \tau_{up}^{av} = \frac{\langle\tau_{up}\rangle}{\langle\tau_{up}\rangle + \langle\tau_{down}\rangle}$$

Reduces to

$$\phi_E(x, \langle\rho\rangle) = c(x)[1 - \text{exp}(-\frac{V\langle\rho\rangle}{c})],$$

for $\langle\tau_{down}\rangle = 0$ and **not** to $\boxed{\phi_E = \min\{c, V\rho\}}$

Reason: Stochasticity of μ needed for the statistical independence of the parts.

This is a theory for large time scales and many suppliers.

OUTLOOK

- General distributions for $\mathcal{U}$ and $\mathcal{D}$
- Networks
- Realistic problems
- Stochastic input