

Lagrange Multiplier Theorem For Optimal Control Problems

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Contents

- 1 Problems
- 2 Theoretical investigations
 - Finite-dimensional case
 - Infinite-dimensional case
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The 'first' optimal control problem

An airplane should be moved in shortest time from y_1 to y_2 . Initially and finally it should stand still. The aircraft can be accelerated or decelerate by a force.

Mathematical formulation for the position $y(t)$ of the airplane

$\min T$ subject to	Cost functional
$m y''(t) = u(t),$	State equation
$y(0) = y_1, y(T) = y_2, y'(0) = y'(T) = 0,$	State equation

'Real-world' problems

A body should be heated up to certain temperature. Heating can be applied at every location.

Mathematical formulation for the temperature $y(x, t)$

$\min \int_0^T \int_{\Omega} \frac{\alpha_1}{2} (y(t, x) - \bar{y}(t, x))^2 + \frac{\alpha_2}{2} u^2(t, x) dx dt$	Cost functional
$y_t - \Delta_x y = u \text{ in } \Omega \times (0, T)$	State equation
$y(x, t) = 0 \text{ auf } \Gamma \times (0, T)$	State equation
$y(x, 0) = 0 \text{ in } \Omega$	State equation

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Questions and Answers

$$\min f(u, y) \text{ subject to } h(u, y) = 0$$

- Existence of an optimal control u ?
- Equation for computing u ?

Reduction of the problem

$f(u, y) : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ is the cost functional

$$\left(\int y^2 + u^2 dx dt \right)$$

$h(u, y) = Bu - Ay$ is the state equation

$$\left(y_t - \Delta y - u = 0 \right)$$

$$\min_{u, y} f(u, y) \text{ s.t. } h(u, y) = Bu - Ay = 0$$

Additional assumption: A regular

(Solvability of pde)

$$\rightarrow y = A^{-1}Bu$$

$$(y = (\partial_t - \Delta)^{-1}u)$$

Yields reduced cost problem with reduced cost functional \tilde{f}

$$\min_u \tilde{f}(u) \text{ where } \tilde{f}(u) = f(u, A^{-1}Bu)$$

Existence

$$\min \tilde{f}(u) \text{ where } \tilde{f}(u) = f(u, A^{-1}Bu)$$

Weierstrass. \tilde{f} continuous, $u \in U \subset \mathbb{R}^m \neq \emptyset$, U bounded and closed, then there exists one optimal control u .

Modification: U unbounded, then consider the cost functional by $\tilde{f}(u) + \lambda \|u\|_2^2$ with $\lambda > 0$. (Tychonoff regularization).

Equation for u (KKT System)

$$\min f(u, y) \text{ subject to } h(u, y) = Bu - Ay$$

Karush–Kuhn–Tucker. f, h continuously differentiable, u^*, y^* the local minimizer under the constraint $h(u, y) = 0$. h satisfies a constraint qualification (e.g. Dh has maximal row rank), then there exists multiplier λ^* , such that

$$Bu^* = Ay^* \quad \text{State equation}$$

$$\nabla_y f(u^*, y^*) + A^T \lambda^* = 0 \quad \text{Adjoint equation}$$

$$\nabla_u f(u^*, y^*) - B^T \lambda^* = 0 \quad \text{Optimality condition}$$

Example

- Optimal control of a diffusion equation in $\Omega \subset \mathbb{R}^n$ with $y = 0$ on $\partial\Omega$ and distributed control

$$-\Delta_x y = u$$

- PDE is well-defined for $y \in X := H_0^1(\Omega)$, $\dim X = \infty$ and reads

$$\int_{\Omega} \nabla_x y \cdot \nabla_x \phi \, dx = \int_{\Omega} u \phi \, dx \quad \forall \phi \in H_0^1(\Omega)$$

- Define $\langle a(y), \phi \rangle := \int_{\Omega} \nabla_x y \cdot \nabla_x \phi \, dx \in H_0^{-1}(\Omega)$ for fixed $y \in H_0^1(\Omega)$

- Constraint is an operator equation

$$a(y) = b(u) \text{ in } (H_0^1(\Omega))'$$

Example (cont'd)

$$\min_{y \in H^1, u \in L^2} \int_{\Omega} (y - y_d)^2 + u^2 dx \text{ subj to } a(y) = b(u) \in H^{-1}$$

Translate concepts of finite-dimensional case

- Continuous cost function \rightarrow Weak semi lowercontinuity
- Multiplier λ and scalar product \rightarrow Element of dual and duality product
- DH has full row rank \rightarrow Surjective operator $DH()$

Existence for infinite dimensional, non-compact sets U

Let $u \in U \subset X \neq \emptyset$ and U bounded, closed and convex. Let $S : X \rightarrow Y$ be a bounded, linear operator and $f(u, Su)$ weak lower semi-continuous and bounded from below. Then, there exists at least one optimal control u .

U bounded can be skipped, if the control is regularized.

Example is bounded from below and weak lower-semi continuous

$$f(u, y) = \|y - \bar{y}\|^2 + \lambda \|u\|^2 = \int_{\Omega} (y - \bar{y})^2 dx + \lambda \int_{\Omega} u^2 dx,$$

Lagrange multiplier theorem in infinite-dimensional spaces

Let X, Y be Hilbert spaces, $f : X \rightarrow \mathbb{R}$ and $H : X \rightarrow Y$ continuously Fréchet differentiable. x^* is the local minimum of f with constraint $H(x) = 0 \in Y$ and $DH(x^*) \in L(X, Y)$ is surjective.

Then, there exists a $\lambda' \in Y'$ such that

$$f'(x^*) - \langle \lambda^*, H'(x^*) \rangle = 0 \quad \text{and} \quad H(x^*) = 0.$$

- Equation in X' , i.e., $\forall x \in X : \langle f'(x^*), x \rangle - \langle \lambda^*, H'(x^*)x \rangle = 0$.
- Finite-dimensional: Each component of $Df + \lambda^T DH$ has to vanish

Reference: Luenburger, 1973

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Sample problem

Control $u(t)$, initial data $\bar{\rho}_1$ and desired state $\bar{\rho}_2$ for $t \in (0, T)$ and $x \in (0, 1)$.

$$\min_{T,u} \int_0^T dt + \int_0^1 \frac{1}{2} (\rho(x, T) - \bar{\rho}_2)^2 dx, \text{ subject to}$$
$$\begin{aligned}\partial_t \rho + \partial_x \rho &= 0, \\ \rho(x, 0) &= \bar{\rho}_1, \\ \rho(0, t) &= u(t)\end{aligned}$$

Lagrangian for this problem

Lagrangian:

$$L(\rho, u, T, \lambda) := \int_0^T dt + \int_0^1 \frac{1}{2} (\rho(x, T) - \bar{\rho}_2)^2 dx + \int_0^1 \int_0^T (-\partial_t \lambda - \partial_x \lambda) \rho dx dt$$

Stationary points:

$$\partial_u L = 0 : \implies \int_0^T \lambda(0, t) \delta u(t) dt = 0 \implies \lambda(0, t) = 0, t \in (0, T)$$

$$\partial_\rho L = 0 : \implies$$

$$-\partial_t \lambda - \partial_x \lambda = 0, \lambda(1, t) = 0, \lambda(x, T) + (\rho(x, T) - \bar{\rho}_2) = 0$$

$$\partial_\lambda L = 0 : \implies$$

$$\partial_t \rho + \partial_x \rho = 0, \rho(0, t) = u(t), \rho(x, 0) = \bar{\rho}_1.$$

Solution to fixed T

Case 1. $T > 1$. Then, $\lambda(0, t) = 0$ for $t \in (0, T)$ and $\lambda(x, T) = \rho(x, T) - \bar{\rho}_2$ for $x \in (0, 1)$ implies that in fact $\rho(x, T) = \bar{\rho}_2$ for $x \in (0, 1)$. Then, $\lambda(x, t) = 0$ for all $(x, t) \in (0, 1) \times (0, T)$. Furthermore, $\rho(x, t) = \bar{\rho}_2$ for $x \in (0, 1)$ implies $u(t) = \bar{\rho}_2$ for $T \geq t \geq T - 1$. In particular, there is no condition on $u(t)$ for $t < T - 1$.

Case 2. $T \leq 1$. Then, $\lambda(0, t) = 0$ for $t \in (0, T)$ and $\lambda(x, T) = \rho(x, T) - \bar{\rho}_2$ for $x \in (0, 1)$ implies that $\rho(x, T) = \bar{\rho}_2$ for $x \in (0, T)$. Then, $u(t) = \bar{\rho}_2$ for $t \in (0, T)$. In particular, $\lambda(x, t) = 0$ for $x \geq t$ and $\lambda(x, t) = 0$ for $x \leq t + (T - 1)$. There is no condition on λ in the remaining strip in the (x, t) -plane of width $T - 1$.

Reduced cost functional $G(T)$

- Definition

$$G(T) = \int_0^T dt + \int_0^1 \frac{1}{2} (\rho(x, T; T) - \bar{\rho}_2)^2 dx$$

where $(u(x, t; T), \rho(x, t; T), \lambda(x, t; T))$

- Values

$$G(T) = \begin{cases} T & T \geq 1 \\ T + (1 - T)(\bar{\rho}_2 - \bar{\rho}_1) & T \leq 1 \end{cases}$$

- Formal derivative

$$G'(T) = H(T - 1) + (\bar{\rho}_2 - \bar{\rho}_1)H(1 - T)$$

- Interpretation for the stationary point of $\partial_T L$?

Open questions

- Model of Armbruster and Ringhofer

$$\partial_t \rho + \partial_x v(\rho) \rho = 0$$

is the model equation for $v(\rho) = 1 - \int \rho dx$

- Differentiation of $v(\rho)$ with respect to ρ ?
- Nonlinear dynamics or just $\partial_t \rho + \partial_x v(t) \rho = 0$

Thank you for your attention.