Charged particle models for the simulation of production networks.

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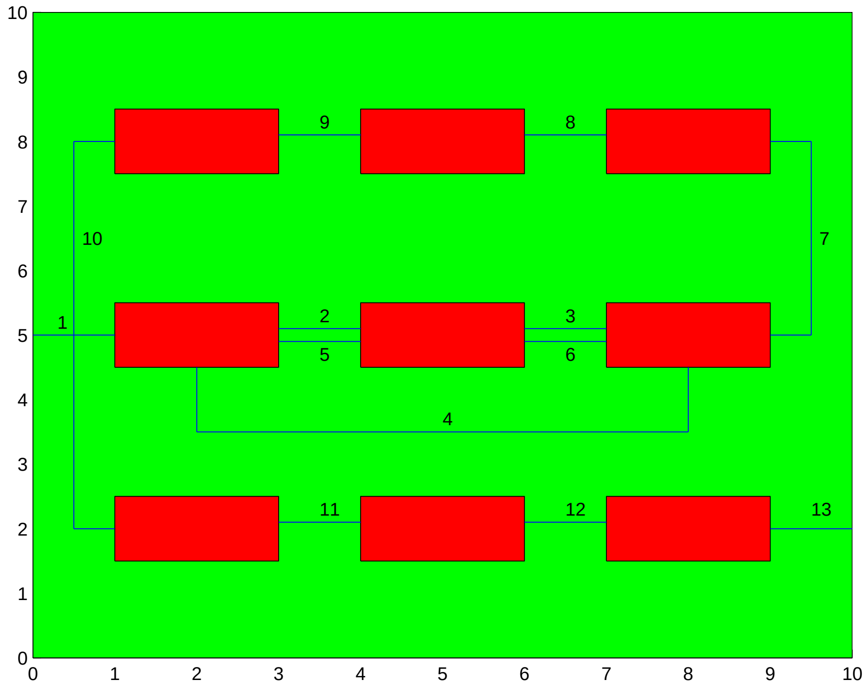
INTRODUCTION



Supply networks:

Consider a network of suppliers with inventories. Items wait their turn in the inventory, get processed and proceed to the next supplier. Applications: Factory floors or any large production networks.

Standard models

1. **Discrete event simulation:** (Equivalent to DSMC for fluids) Each client is modelled individually (according to random distributions for wait and processing times). Advantage: Most accurate. Disadvantage: Computational effort proportional to the number of parts.

2. **Queueing theory:** Derive equations for the means (and some time the variance) of the resulting fluxes on large time scales, assuming arrivals and processing times are distributed according to certain distributions. Yields steady state results for queue lengths etc.)

3. Fluid models: ODE models for the 'Work in Progress' (WIP) of processor and buffer number m:

$\frac{d}{dt}W_m = F_m^{in} - F_m^{out}$. Control or model $F_m^{in/out}$ according to queuing theory.

Advantage: Scalable (work with densities). Disadvantage: Assume in general an infinitesimal processing time. Always assume quasi steady states.

Goal: Treat large queueing systems (in the first iteration deterministically) in non - equilibrium situations using fluid dynamic and kinetic models.

Tool: Kinetic and fluid dynamic models. Model parts as particles advancing in the system + continuum limit.

Advantages: Scalable for a large number of parts (densities). Represent transient behavior better than fluid models.

Charged particle models:

Detailed modelling of particle behavior; $\longrightarrow$ kinetic equations as continuum models; $\longrightarrow$ fluid models for macroscopic item densities. Analogy to traffic flow models.

Charged particle models for supply chains

Introduce a stage variable $x \in [0,1]$. Parts enter at stage $x = 0$ and exit at stage $x = 1$.

One supplier corresponds to an interval of length Δx.

Parts move along trajectories $x = \xi_n(t)$ with velocity u according to the Newton equations.

$$\frac{d}{dt}\xi_n(t) = u(\xi_n, f),\ n = 1, .., N$$

u : velocity profile, dependent on the state f of the density f of the whole ensemble.

Conservation law:

$$\partial_t f + \partial_x[u(x,f)f] = 0$$

OUTLINE$_6$

- Policies:

  $\longrightarrow$ Kinetic models with restricted flux.

  $\longrightarrow$ Moment closures and fluid models: Conservation laws with distributional solutions.

- Stochasticity:

  $\longrightarrow$ Random capacities.

  $\longrightarrow$ Collisions and the Boltzmann equation.

Flux restricted kinetic models (policies)

(ADR, submitted , SIAP, TTSP '05)

Suppliers have a limited capacity $c(x)$. Leads to kinetic models and their moment equations with discontinuities in the fluxes because of a velocity cutoff.

In order to model policies, we give each item a priority p, leading to

$$\partial_t f(x,p,t) + \partial_x [u(x,p,f)f] = 0$$

Items are processed in order of their priority p until the flux reaches c:

$$u(x,p,f) = \begin{matrix} v(x) & \text{for} & p > p_0 \\ 0 & \text{for} & p < p_0 \end{matrix} = v(x)H(p-p_0)$$

The velocity cutoff $p_0 = p_0(x, f)$ has to be chosen such that the actual flux F satisfies

$$F(x,t) := \int u(x,p,f) f(x,p)\, dp = v(x) \int_{p_0}^{\infty} f(x,p)\, dp \leq c(x)$$

holds. Defining the unrestricted flux

$$\phi_f(x,p,t) = v(x) \int_p^{\infty} f(x,q,t)\, dq,$$

we have

$$p_0 = \begin{pmatrix} -\infty & \text{for} & \phi_f(x,-\infty,t) < c(x) \\ \phi_f(x,p_0,t) = c(x) & \text{for} & \phi_f(x,-\infty,t) > c(x) \end{pmatrix}$$

this yields $p_0 = \phi_f^{-1}(x,c,t)$ and

$$H(p - p_0) = H(c - \phi_f(x,p,t))$$

The flux limited model

$$\partial_t f(x,p,t) + \partial_x[vH(c-\phi_f)f] = 0, \quad \phi_f(x,p,t) = v\int_p^\infty f(x,q,t)\,dq$$

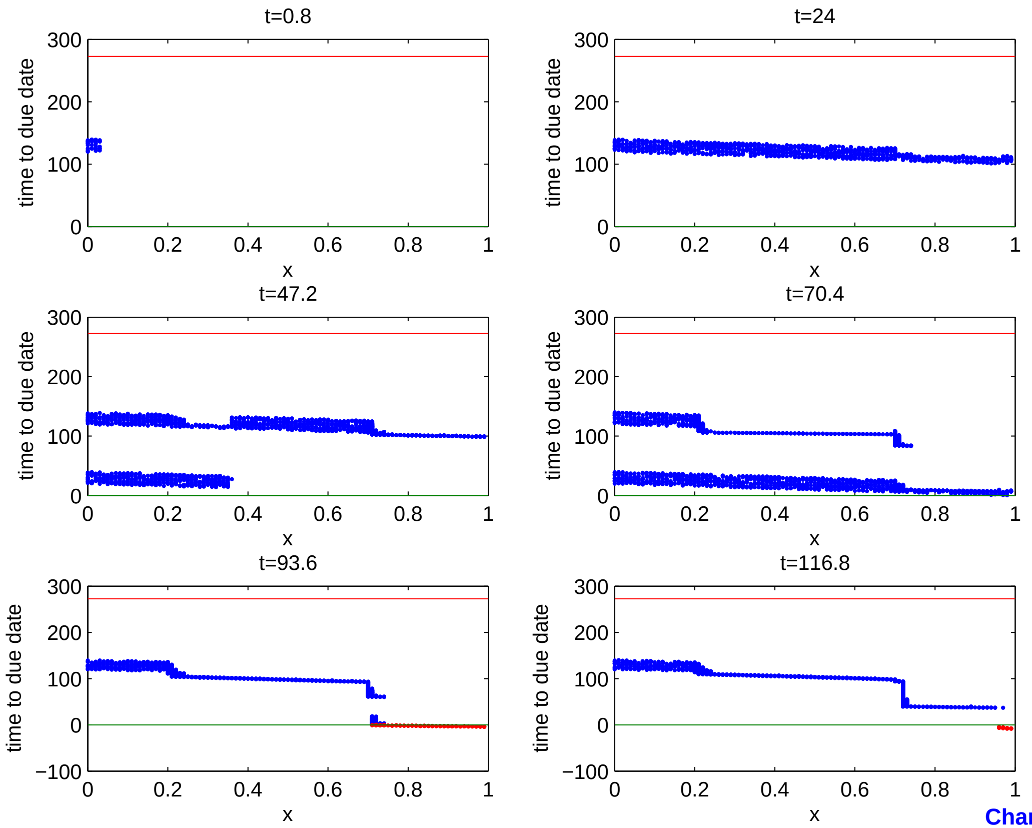
Remark: Policies could be changed either continuously or randomly

$$\partial_t f(x,p,t) + \partial_x[vH(c-\phi_f)f] + E\nabla_p f = Q[f],$$

Example: THE HOT LOT PROBLEM$_{11}$

Production system in equilibrium. Hot lots (particles with highest priority arrive) receiving absolute priority.

Q: How long does it take to recover equilibrium?



20 production steps.

Bottlenecks at steps 5 and 15.

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Hot lots arrive at $t = 40$

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Bottleneck 5 can support one flow but not both.

Bottleneck 15 cannot support single flow.

MOMENT CLOSURES $_{16}$

$\longrightarrow$ Replace kinetic model by macroscopic fluid equations for the moments.

$\longrightarrow$ Leads to conservation laws with distributional solutions.

- Multiphase equations.
- Exact zero order moment closure.

Multiphase closures

Assume the $f(x,p,t)$ is supported by maximally N $\delta-$ functions for any given x. Close the corresponding moment system. (Jin '02). Not more than N particles can pass each other at any given time.

Ansatz: Y_n: attribute of phase number n, ρ_n: density of phase number n.

$$f(x,p,t) = \sum_{n=1}^{N} \rho_n(x,p,t)\delta(p - Y_n(x,t))$$

$$\partial_t m_j + \partial_x F_j = q_j,\ j = 1,..,2N$$

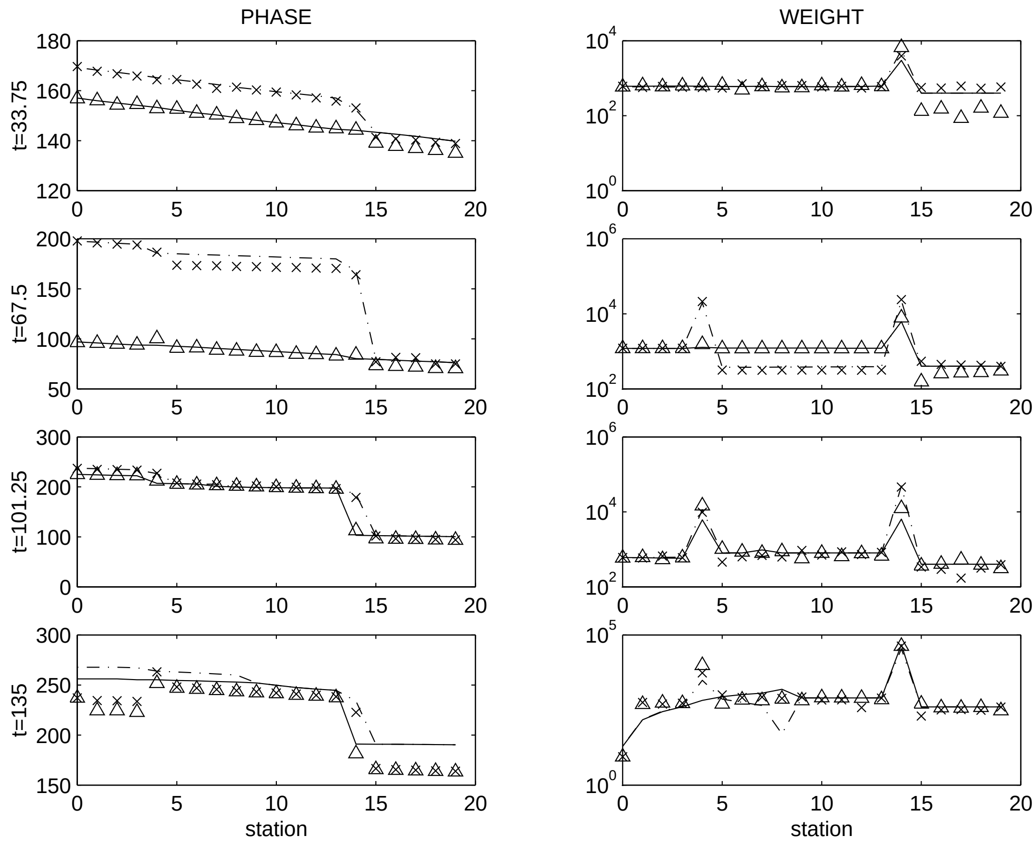
Closure by expressing moments and fluxes in terms of ρ_j and Y_j

$$m_j = \int p^j f(x,p,t)\, dp = \sum_{n=1}^{N} \rho_n Y_n^j,$$

$$F_j = v(x,t) \sum_{n=1}^{N} Y_N^j H(c(x,t) - \phi_f(x,Y_n,t))\rho_n$$

$$\phi_f(x,Y_n,t) = v(x,t) \sum_{m=1}^{N} H(Y_n - Y_m)\rho_m$$

Comparison$_{20}$



Compute two - phase solution and two phase approximation from the kinetic solution.

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left=Y, right=ρ

$-, -.-$ hot and cold phase from two phase model.

$\triangle, \times$ kinetic model

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perfect match: $\triangle = -$ and $\times = -$

The zero order moment model

If we are not interested in the distribution of items in p, we can only consider a moment model for $\rho(x,t) = \int f(x,p,t)\,dp$.

$$\partial_t f(x,p,t) + \partial_x[vH(c - v\phi_f)f] = 0, \quad \phi_f(x,p,t) = v\int_p^\infty f(x,q,t)\,dq$$

Since the flux depends on p only through ϕ_f, this can be integrated exactly! Note that $\partial_p \phi_f = -vf$ holds.

$$\partial_t \phi_f(x,p,t) = -v\partial_x \int_p^\infty H(c - \phi_f)\partial_{p'}\phi_f\,dp', \quad = -\partial_x \int_{\phi_f}^0 vH(c - v\phi)\,d\phi$$

The total item density $\rho(x,t) = \int f(x,p,t)\,dp = \phi_f(x,-\infty,t)$ satisfies the conservation law

$$\partial_t \rho + \partial_x F = 0, \quad F(x,\rho) = \min\{c(x), v\rho\}$$

Analytical properties of the flux restricted fluid model$_{23}$

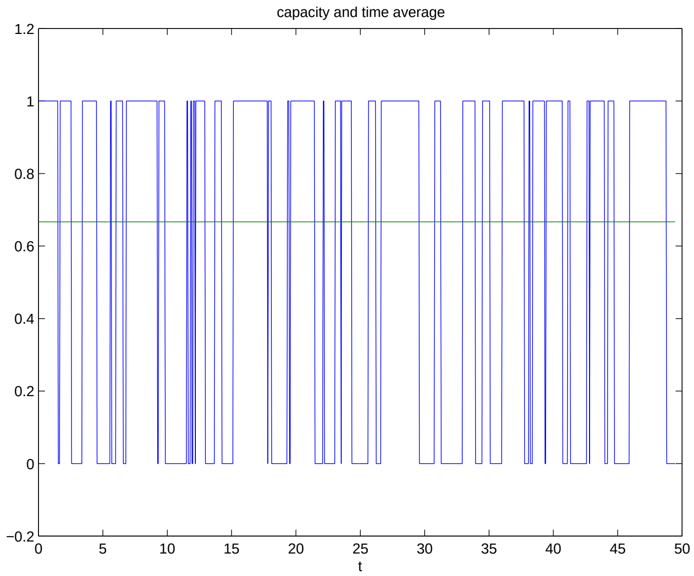
- Since information flows strictly from left to right the flux F can become discontinuous whenever c is discontinuous.
- The density ρ can exhibit $\delta-$ function concentrations at points where there is a sharp drop in the local capacity $c(x)$ of the processor chain.
- These correspond to a buildup of queues in the supply chain.

PART II - Stochasticity$_{25}$

- Fluid models assume that the supply chain works like an automaton.
- Realistically, the capacity c as weall as the arrivals (influx) has to be modelled as a random variable.
- **Goal:** Include the stochasticity into the flux restricted model.
- Replace c by $c(t)$, a non - Markovian stochastic process.

Consider only one supplier $c = c(t)$;

either active or inactive $(c(t) = 1$ or $c(t) = 0)$.



1. Start with $c(0) = 1$.
2. Choose a random τ_{up} and $c(t) = 1$ for $0 < t < \tau_{up}$
3. Choose τ_{down} and $c(t) = 0$ for $\tau_{up} < t < \tau_{up} + \tau_{down}$.
4. Repeat.

Question

Is it permissible to simply replace $c(x,t)$ by $\langle c\rangle(x) = \frac{\langle\tau_{up}\rangle}{\langle\tau_{up}\rangle+\langle\tau_{down}\rangle}$ in the deterministic model

$$\partial_t\rho + \partial_x \min\{\langle c\rangle, v\rho\} = 0$$

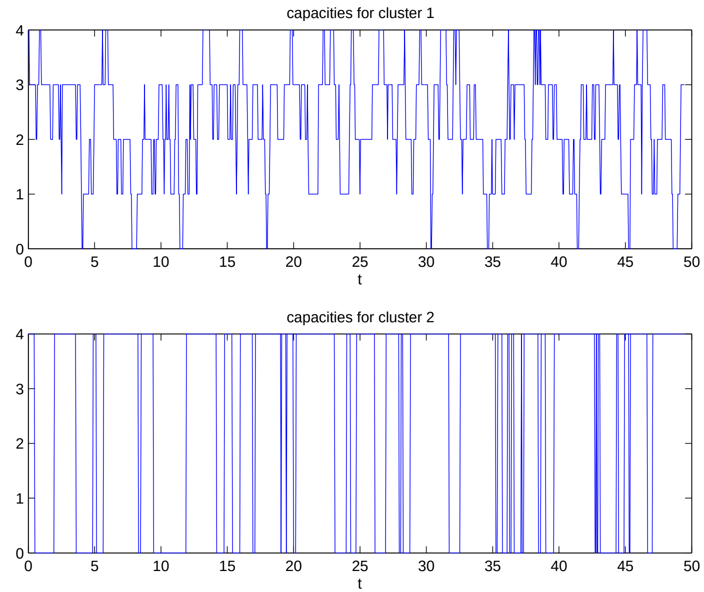
at least in the long time average?

Answer: No.

Conservation law has to be modified using a detailed description of the stochastic process.

Leads to a Boltzmann type model for probability densities, similar to charged particle transport under strong external forces.

Example$_{28}$



Consider a chain of two suppliers S_1 and S_2.

- S_1 consists of 4 nodes with capacity 1,
- S_2 of one node with capacity 4.

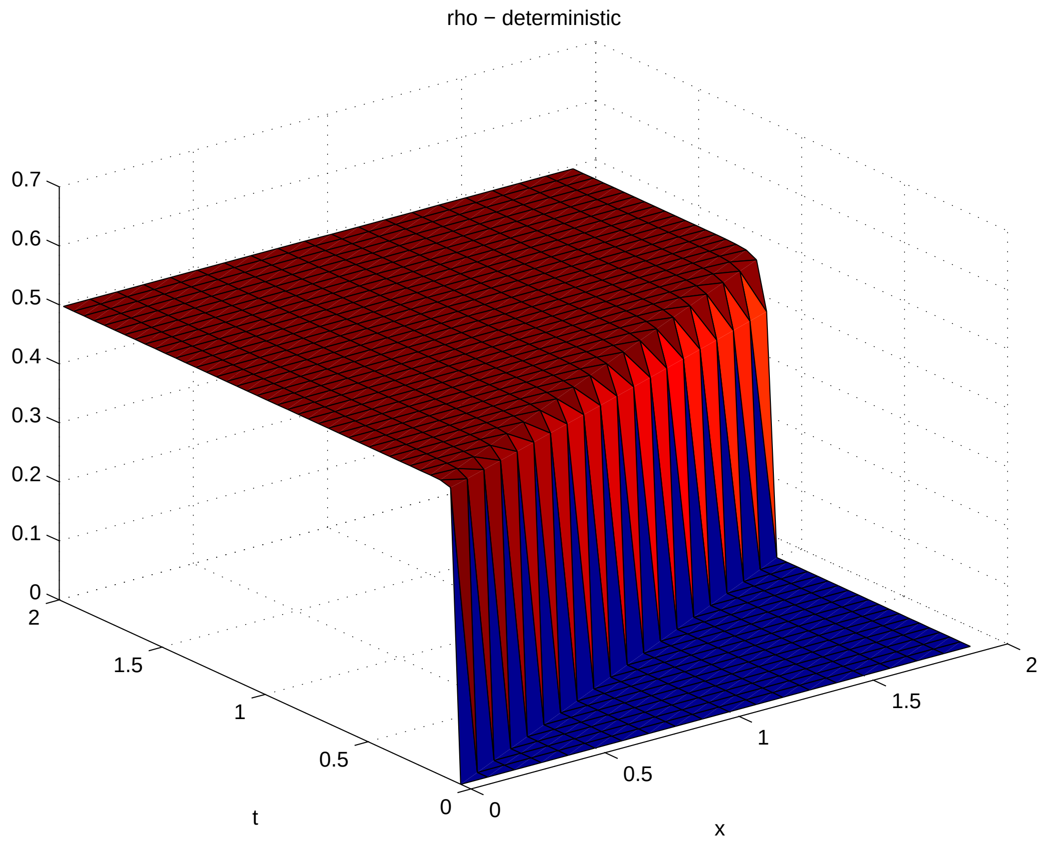
Capacities:

$$c_1(t) = \sum_{n=1}^{4} \gamma_{1n}(t), \quad c_1(t) = 4\gamma_2(t), \quad \langle c_{1,2} \rangle = \frac{4\langle \tau_{up} \rangle}{\langle \tau_{up} \rangle + \langle \tau_{down} \rangle}$$

S_1 and S_2 are, under the expectation, identical!

The deterministic solution

$$\partial_t \rho + \partial_x \min\{\langle c_{1,2}\rangle, v\rho\} = 0$$

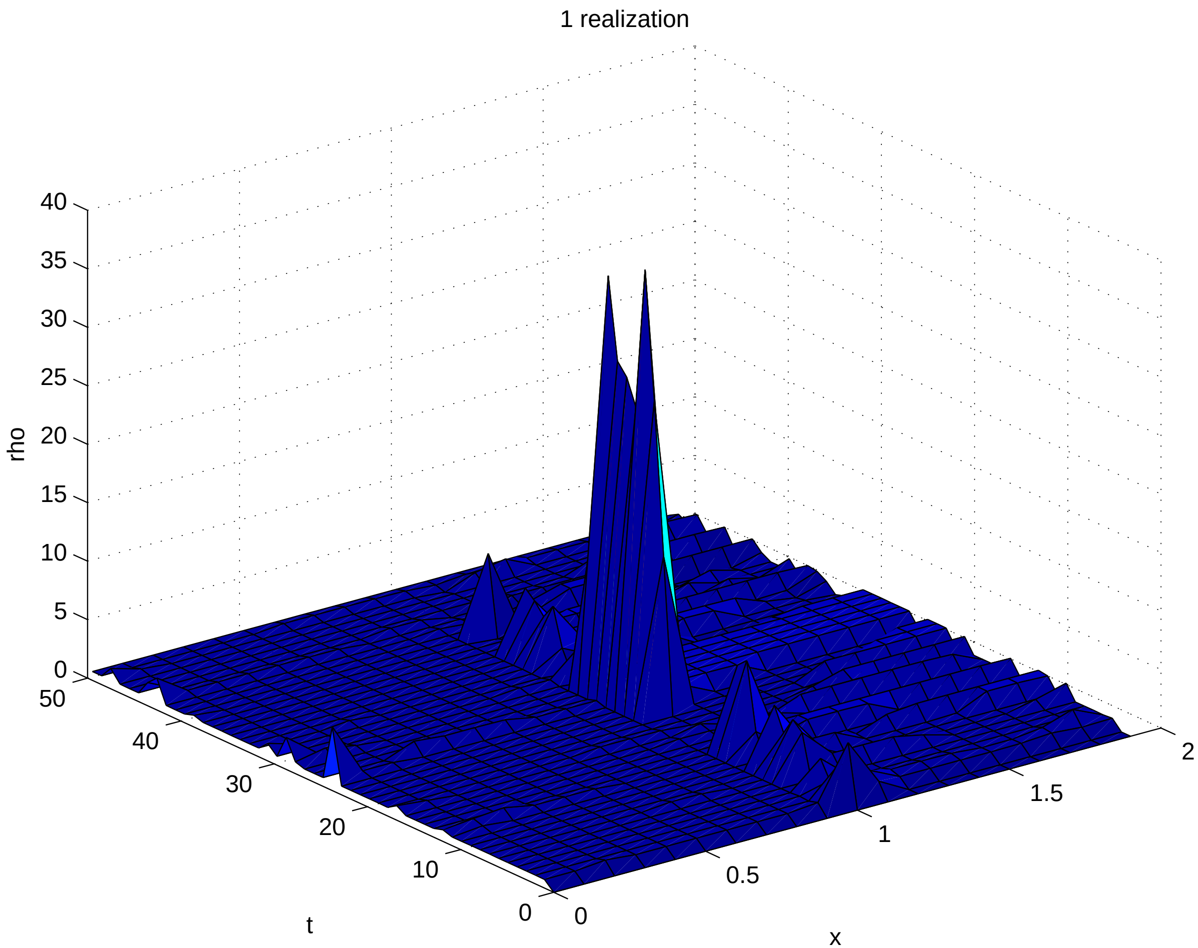


Influx below $\langle c_{12}\rangle$

$\Rightarrow\ \partial_t \rho + v\partial_x \rho = 0$

One realization of the stochastic equation

$$\partial_t \rho + \partial_x \min\{c(x,t), v\rho\} = 0$$



Temporary buildup of $\delta-$

functions when $c(x,t) = 0$

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even for influx $< \langle c_{12} \rangle$

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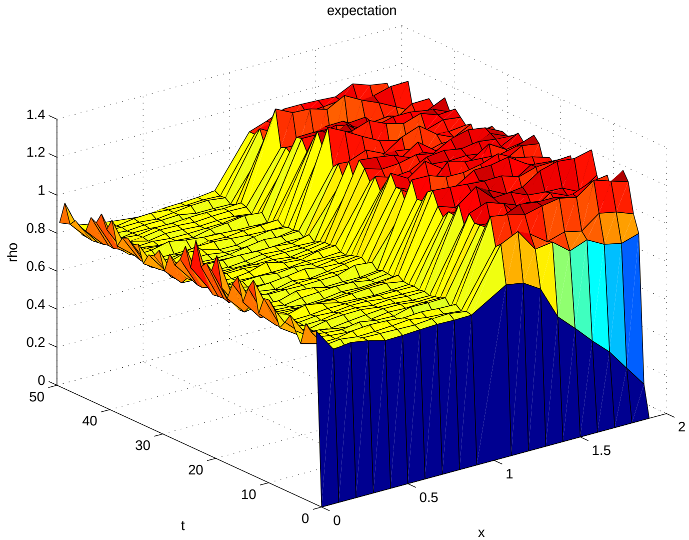
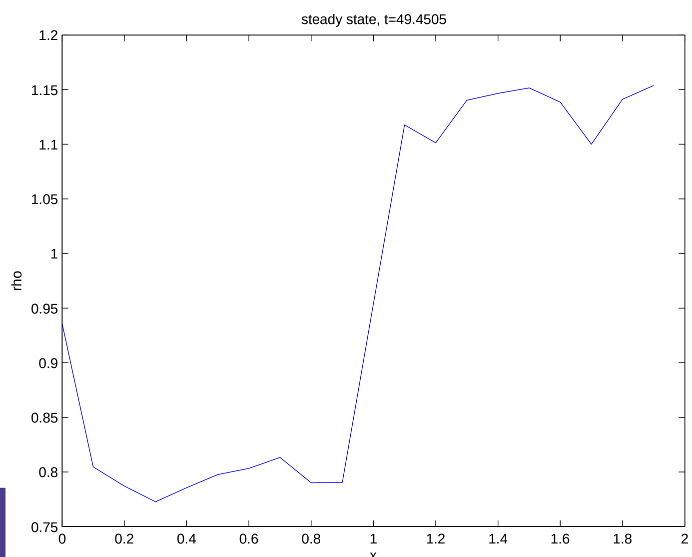
S_1 less random

Expectation





Average over 100 realizations.

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$\delta-$ functions average out.

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S_1 and S_2 remain different!

Steady state:

Piecewise constant.

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The model for capacities and elapsed times $_{30}$

$c(t)$ is a non - Markovian process. $\frac{d\mathcal{P}}{d\gamma}\{c(t)=\gamma\}$ will depend on how long the the supplier has been in this state.

$\tau(t)$: the time since the last capacity change has to be an independent variable.

A Boltzmann model - Capacity switch as collisions

r: random variable (0 or 1) deciding on whether to switch capacities

$$\mathcal{P}\{r=1\} = \Delta t\omega(c,\tau), \quad \mathcal{P}\{r=0\} = 1-\Delta t\omega(c,\tau),$$

$\omega(c,\tau)$: switching frequency

If $r=0$: $c(t+\Delta t)=c(t), \quad \tau(t+\Delta t)=\tau(t)+\Delta t$

If $r=1$: $c(t+\Delta t)=c_{new}, \quad \tau(t+\Delta t)=0$

$d\mathcal{P}\{c_{new}=\gamma\}=S(\gamma,c(t))\,d\gamma$

Proposition: In the limit $\Delta t\to 0$ the probability density $f(c,\tau,t)\,dc\tau$ satisfies

$$\partial_t f+\partial_\tau f=\delta(\tau)\int S(c,c')\omega(c',\tau')f(c',\tau',t)\,dc'\tau'-\omega(c,\tau)f$$

How to choose the frequency $\omega(c,\tau)$ 33

Relate ω to the probability of the supplier being active or inactive.

Equivalent to the definition of a mean free path in Monte carlo methods (Hockney & Eastwood).

The distribution $q(c,T)$ of the time between switches is given by

$$q(c,T) = \omega(c,T)\exp[-\int_0^T d\tau\,\omega(c,\tau)\,d\tau] \iff$$

$$\omega(c,\tau) = \frac{q(c,\tau)}{1-\int_0^\tau q(c,T)\,dT}$$

The joint probability

Since c, τ are independent of the items (but not vice versa), the probabilistic item density $R(x, c, \tau, t)$ of the item being at x while the capacity is c and the elapsed time since the last switch is τ satisfies (under a mean field assumption)

$$\partial_t R(x, c, \tau, t) + \partial_x [u(c, \rho) R] + Q[R] = 0,$$
$$u(c, \rho) := \min\{\tfrac{c}{\rho}, v\}, \quad \rho(x, t) = \int R(x, c, \tau, t)\, dc\tau$$

$$Q[R] = \partial_\tau R + \omega R - \delta(\tau) \int \omega(c', \tau') S(c, c') R(x, c', \tau', t)\, dc'\tau'$$

Remarks

$\longrightarrow$ Q scales with $\frac{1}{\tau}$. For large time scales we have $Q \longrightarrow \frac{1}{\varepsilon}Q$ with $\varepsilon = \frac{\langle\tau\rangle_q}{t_{scale}}$

$\longrightarrow$ This gives rise to fluid equations for $\rho(x,t) := \int R(x,c,\tau,t)\,dc\tau$ via a Chapman - Enskog expansion.

$\longrightarrow$ Therefore, in zero' th order, the probabilistic version of the conservation law is of the form

$$\partial_t \rho + \partial_x F = 0, \quad F = \int u(\rho,c,f) R_{eq}(x,c,\tau)\,dc\tau = \int \min\{c, v\rho\} f_{eq}(c,\tau)\,dc\tau$$

with f_{eq} the kernel of Q, i.e. $Q[f_{eq}] = 0$

$\Rightarrow$

The difference in the example above results from

$$\int \min\{c, v\rho\} f_{eq}(c,\tau)\, dc\tau \neq \min\{\langle c \rangle, v\rho\}$$

even if $v\rho < \langle c \rangle$ holds!

The collision operator Q[f] $_{36}$

$\longrightarrow$ The stochastic case is determined by analyzing the kernel of $Q[f]$

$$Q[f] = \partial_\tau f + \omega(c,\tau) f - \delta(\tau) \int \omega(c',\tau') S(c,c') f(c',\tau',t)\, dc'\tau'$$

$\longrightarrow$ This is structurally the same collision operator as used for the scattering of charged particles with a background in the case when the electric field balances the collisions. (Poupaud, Cercignani, Gamba, Levermore)

$$\partial_t f + v \cdot \nabla_x f + \frac{1}{\varepsilon}[E \cdot \nabla_v f + f - M(v) \int f\, dv'] = 0$$

$\longrightarrow$ Yields equilibrium solutions with nonzero mean velocity and therefore to a hydrodynamic (hyperbolic) limit with a small diffusive correction.

N sub-nodes

In the case of a supplier consisting of N sub - nodes the capacities can only take on integer values from $n = 0$ to $n = N$.

$$f(c,\tau) = \sum_{n=0}^{N} f_n(\tau)\delta(c-n),$$

$\Rightarrow$ Q becomes a matrix operator in τ.

$$Q[f]_n = \partial_\tau f_n + \omega_n(\tau) f_n - \delta(\tau) \sum_{k=0}^{N} \int S_{nk}\omega_k(\tau') f_k(\tau', t)\, d\tau'$$

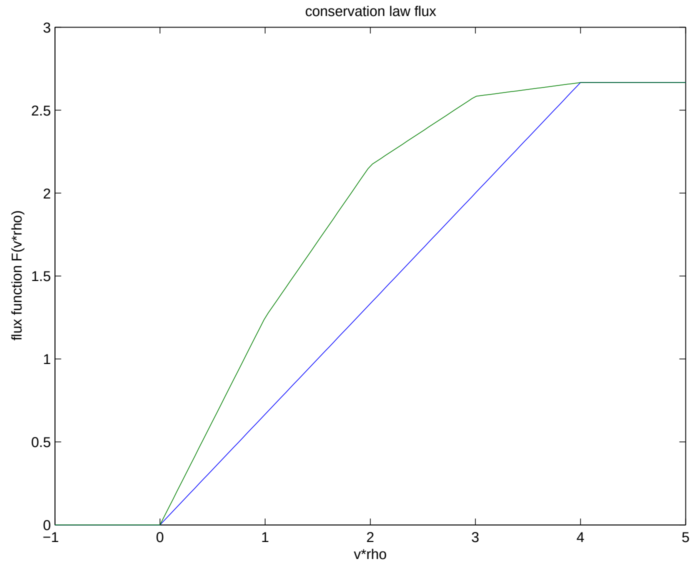
$$\int f_{eq}(c,\tau)\, d\tau = \sum_{n=0}^{N} \gamma_n(\tau)\delta(c-n),$$

giving in zero'th order the conservation law for the stochastic system

$$\partial_t \rho + \partial_x F = 0, \quad F = \sum_{n=0}^{N} \langle\gamma_n\rangle \min\{n, v\rho\}$$

$$\langle\gamma_n\rangle = 2^{1-N} \binom{N}{n} \frac{n\langle\tau_{up}\rangle + (N-n)\langle\tau_{down}\rangle}{N(\langle\tau_{up}\rangle + \langle\tau_{down}\rangle)}$$

The zero order flux $_{40}$



$F(\rho)$ is piecewise linear, consisting of N segments.

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The maximum value $\frac{\langle\tau_{up}\rangle}{\langle\tau_{up}\rangle+\langle\tau_{down}\rangle}$ is reached at$v\rho = N$

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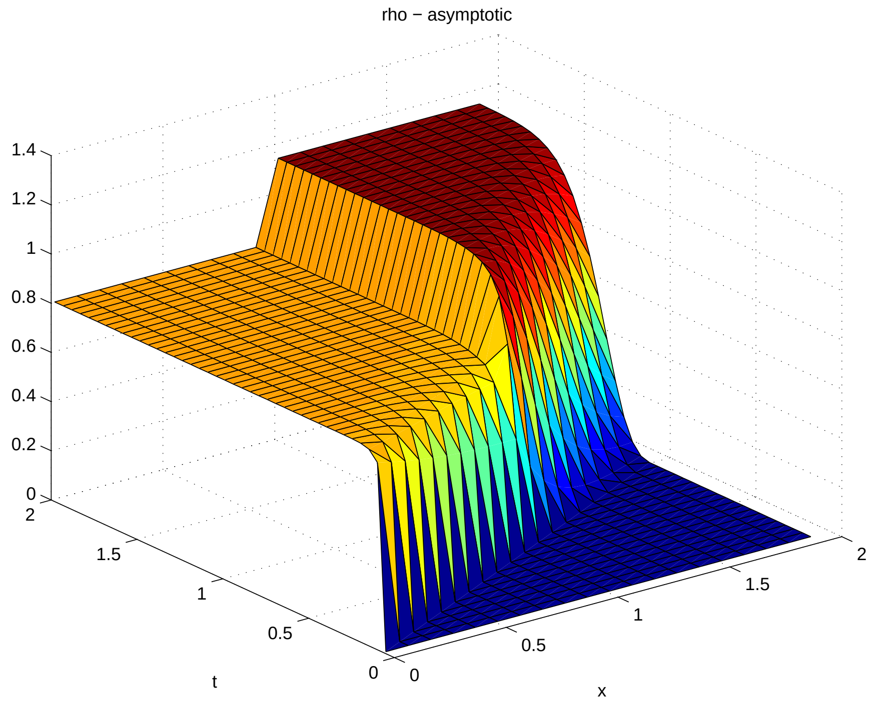
Using a larger number of sub - nodes increases the expected flux.

Qualitatively the same as simple traffic flow (Greenberg ’70s)

$$F = F_0 - \ln(\frac{\rho+\rho_0}{\rho_0})$$

Asymptotic solution



S_1: four nodes; S_2: one node.

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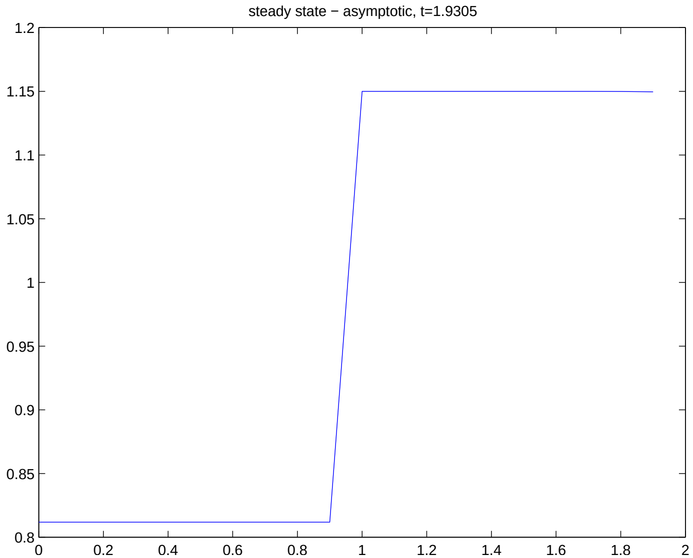
$$\partial_t \rho + \partial_x F = 0$$

Steady state: piecewise constant.

Steady state



Asymptotic steady state.

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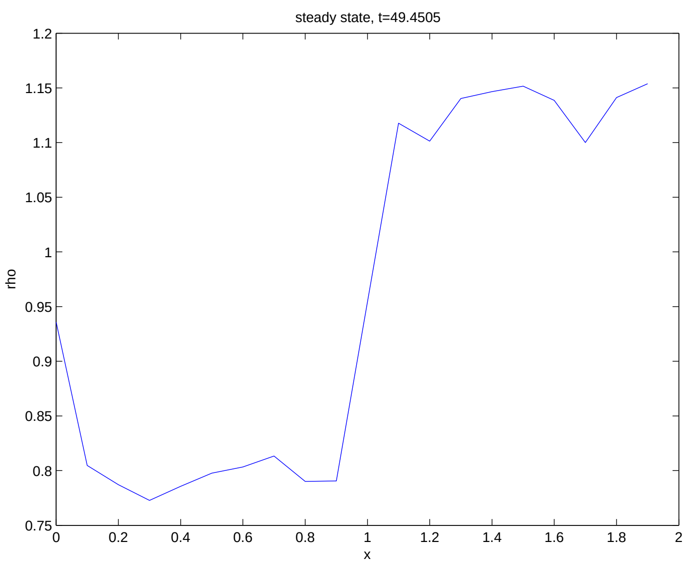

Stochastic steady state

averaged over 100 realizations.

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Conclusions

- Automata kinetic and fluid models for supply chains.
- Flux functions uniformly bounded $\Rightarrow$ discontinuous fluxes and density concentrations.
- Including stochastic capacities adds shocks (structurally similar to simple traffic flow models).
- Dynamic equivalent to the Polaczek - Khintchine formula.
- Still need to add random arrivals (influx)