LECTURE 2-3: Stochasticity

PDF available at http://math.la.asu.edu/∼chris/CIME09/CIME09.htm

CONTENTS

Part 1: Traffic flow models (fluid dynamics)$_{03}$

- Introduce an artificial variable x, the stage of the production process. Parts enter at stage $x = 0$. Finished product leaves at stage $x = 1$. Parts travel on the 'road' from $x = 0 \to x = 1$.
- Traffic models for production networks constitute an intermediate stage between DES and rate equations. PDE's $\Rightarrow$ amenable to analysis and optimization.
- Large body of theory: Lighthill, Whitham, Prigogine, Daganzo,Helbing,Peters, Klar, Rascle

Production Systems vs. Traffic Flow $_{05}$

Conservation Laws:

$$\boxed{\partial_t \rho + \partial_x [\phi(\rho)] = 0}$$

ρ: Work in progress (WIP-) density or density of vehicles

x: stage of the production process. $x = 0$: unfinished product,

$x = X$: finished product, traffic flow: road

ϕ: flux function

Similarities between Traffic and Production Models $_{06}$

- Complexity and Topology: Complex re - entrant production systems. Networks of roads.
- Control: Policies for production systems. Traffic control mechanisms.
- Random behavior.
- First Principle Models: Discrete Event Simulation (DES), Multi - Agent Models (incorporate stochastic behavior) $\Rightarrow$ kinetic equations for densities (mean field theories, large time asymptotics) $\Rightarrow$ fluid dynamics $\Rightarrow$ rate equations (fluid models).
- Simulation $\Rightarrow$ optimization and control.

Differences between Traffic and Production Modeling $_{08}$

- Capacity Limits: Limited capacity of processors (machines), limited space (capacity of the road) $\Rightarrow$ backward wave propagation in traffic flow $\frac{\phi(\rho)}{\rho} > 0$ and $\phi'(\rho) < 0$.
- Parameters and Control Variables:
  - Traffic: randomly given influx and individual behavior, distribute road capacities.
  - Production: randomly given demand, choose start rate and (to some extent) topology.

Example: $M/M/1$ queues and simple traffic flow models $_{10}$

Arrivals and processing times governed by Markov processes:

$$\boxed{v(x,\rho) = \frac{c(x)}{1+\rho}, \quad c(x) = \frac{1}{\langle \text{processing times} \rangle}}$$

$c(x)$: service rate or capacity of the processor at stage x.

Simplest traffic flow model (Lighthill - Whitham - Richards)

$$\boxed{v(x,\rho) = v_0(x)\left(1 - \frac{\rho}{\rho_{jam}}\right)}$$

- In supply chain models the density ρ can become arbitrarily large, whereas in traffic the density is limited by the space on the road ρ_{jam}.

The phase velocity $_{12}$

$$v_{phase} = \frac{\partial}{\partial \rho}[\rho v(x,\rho)]$$

$$\boxed{v_{phase} = \frac{c(x)}{(1+\rho^2)} > 0, \quad v_{phase-traffic} = v_0(x)[1 - \frac{2\rho}{\rho_{max}}]}$$

- In supply chain models the propagation of information (shock speeds) is strictly forward $v_{phase} > 0$, whereas in traffic flow models shock speeds can have both signs.

Problem:

- Queuing theory models are based on quasi - steady state regime. Modern production systems are almost never in steady state. (short product cycles, just in time production).
- Goal: Derive non - equilibrium models from first principles (first for automata) and then including stochastic effects.

CONTENTS

First principle models for automata $_{15}$

- Assume processors work deterministically like automata. A processor located in the infinitesimal stage interval of length Δx needs a time $\tau(x) = \frac{\Delta x}{v_0(x)}$ to process an item.
- It cannot accept more than $c(x)\Delta t$ items per infinitesimal time interval Δt.

N-curves (Newell 1958):

Goal: Find flux function for $\partial_t \rho + \partial_x \phi = 0$

$$\boxed{U(x,t) = \int_{-\infty}^{t} \phi(x,s)\, ds \Rightarrow \rho(x,t) = -\partial_x U}$$

Model the inverse of U w.r.t time:

$$U(x,t) = s \iff Z(x,s) = t$$

$U(x,t)$: Number of parts processed by processor x at time t= Number of the part processed by processor x at time t.

$Z(x,s)$: The time processor x processes part number s.

Modeling Z

- $T_a(x,s)$: arrival time at beginning of queue,
- T_f: time part s is fed into processor x

$$T_f(x,s) = \max\{T_a(x,s),\ T_f(x,s-\Delta s) + \frac{\Delta_s}{c(x)}\}$$

$$T_a(x+\Delta x,s) = T_f(x,s) + \frac{\Delta x}{v_0(x)} = Z(x,s)$$

$$\boxed{Z(x,s) = \max\{Z(x-\Delta x,s) + \tfrac{\Delta x}{v_0(x)},\ Z(x,s-\Delta s) + \tfrac{\Delta s}{c(x)}\}}$$

The inverse relations

$$Z(s,x) = t \iff U(x,t) = s$$

$$\Rightarrow \partial_s Z = \frac{1}{\partial_t U}, \quad \partial_x Z = -\frac{\partial_x U}{\partial_t U}$$

$\Rightarrow$

$$0 = \max\{\Delta x \frac{\partial_x U}{\partial_t U} + \frac{\Delta x}{v_0(x)}, -\Delta s \frac{1}{\partial_t U} + \frac{\Delta s}{c(x)}\}$$

$\Rightarrow$

$$\boxed{\partial_t U = \min\{-v_0 \partial_x U, c\}}$$

$$\partial_t U = \phi, \quad -\partial_x U = \rho, \quad \partial_t \rho + \partial_x \phi = 0$$

$$\boxed{\Rightarrow \partial_t \rho + \partial_x [\min\{c(x), v_0(x)\rho\}]}$$

Summary

The simple deterministic automaton model yields the conservation law

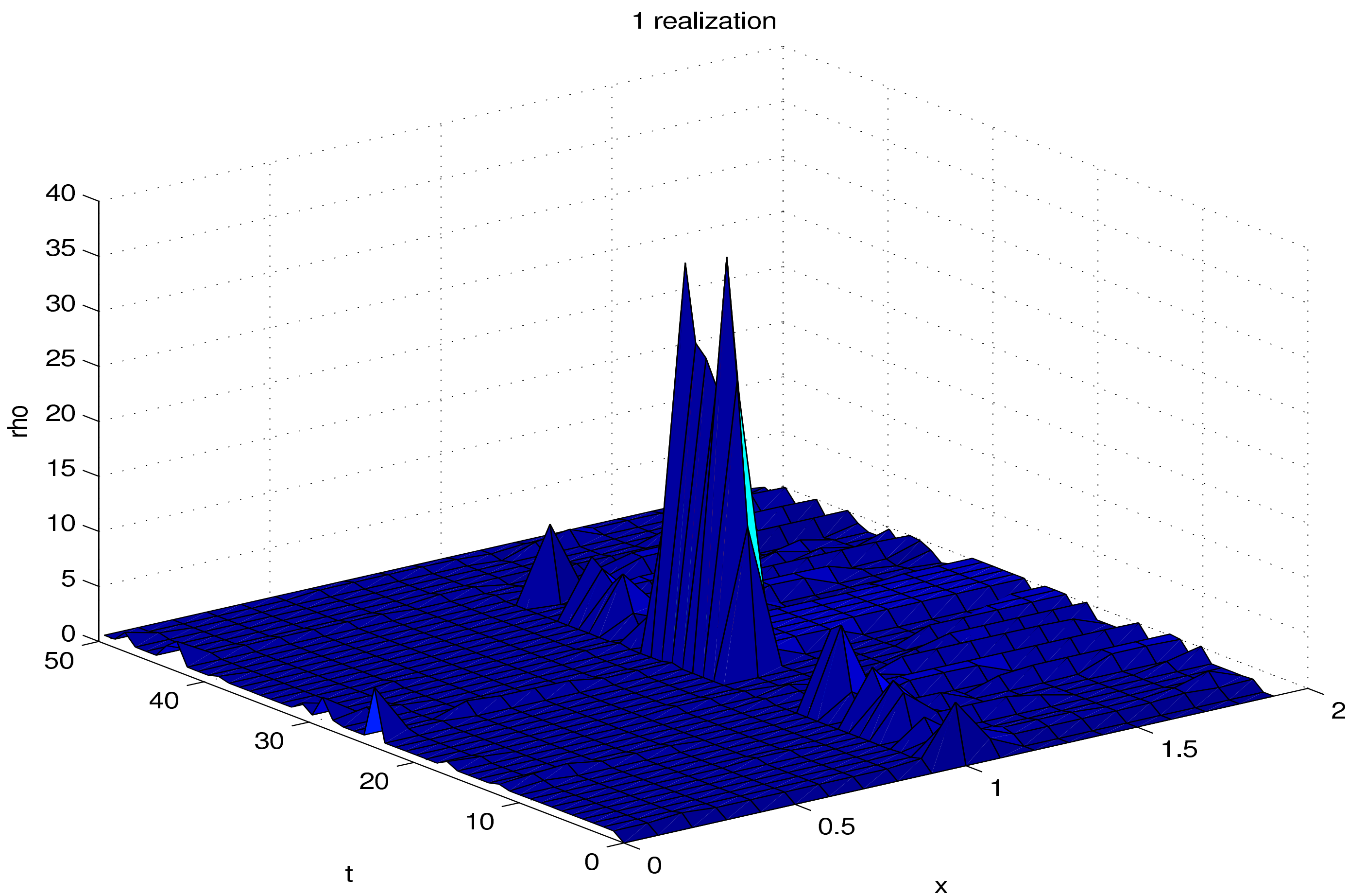
$$\boxed{\Rightarrow \partial_t \rho + \partial_x [\min\{c(x), v_0(x)\rho\}]}$$

Bottlenecks $_{20}$

$$\boxed{\partial_t \rho + \partial_x \phi(x,\rho) = 0, \quad \phi(x,\rho) = \min\{c(x), v_0(x)\rho\}}$$

- No maximum principle (similar to pedestrian traffic with obstacles).
- The capacity $c(x)$ is discontinuous if nodes in the chain form a bottleneck.
- Flux ϕ discontinuous $\Rightarrow$ density ρ distributional. (alternative model by Klar, Herty '04).
- Random server shutdowns $\Rightarrow$ bottlenecks shift stochastically.

A bottleneck in a continuous supply chain

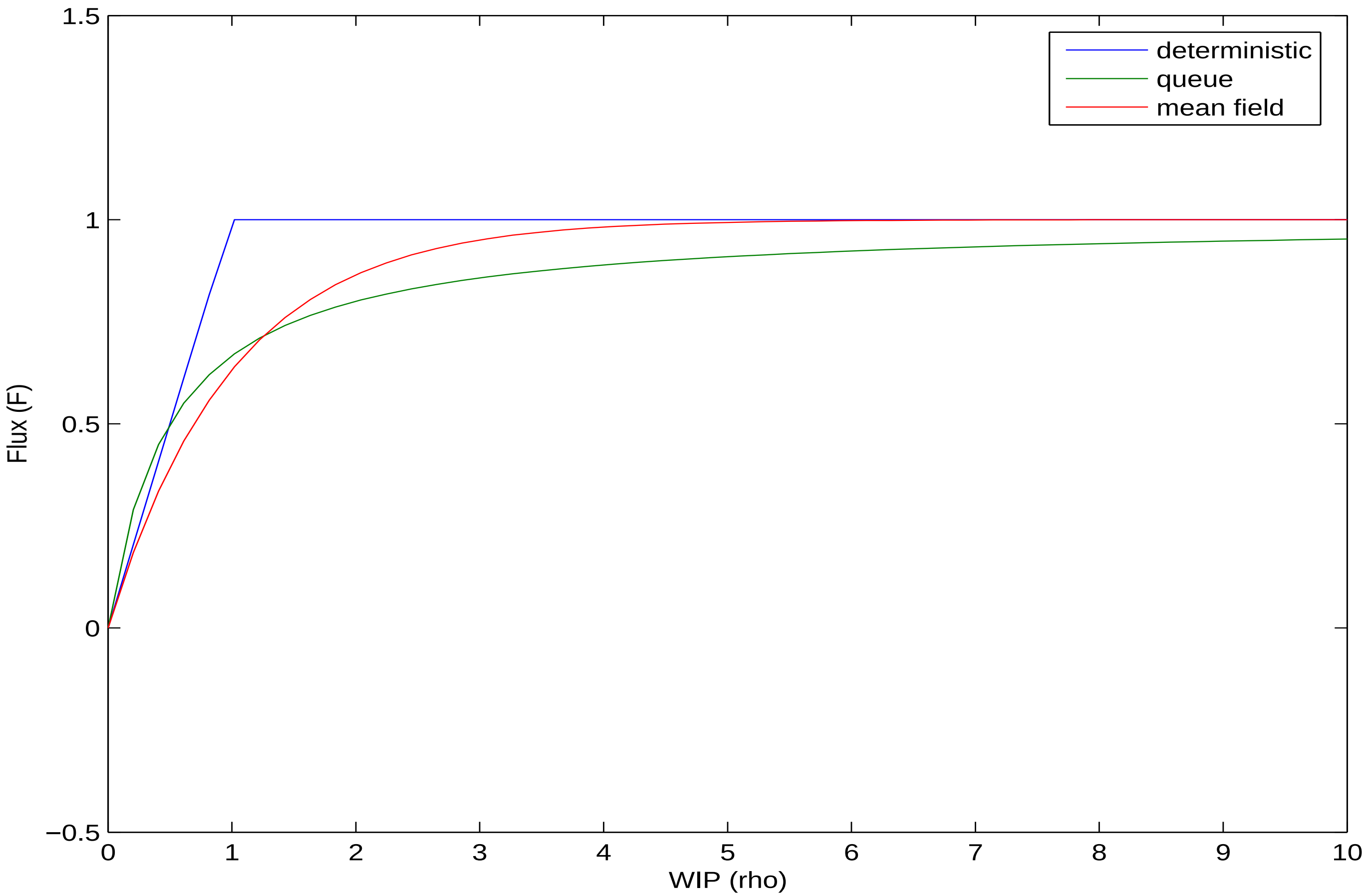


Temporary overload of the bottleneck located at $x = 1$.

Shortcoming of deterministic supply chain models $_{23}$

- Deterministic models produce a linear steady state WIP - flux relation below capacity, and no steady state above capacity.
- Stochastic models produce a diagram with the capacity as an asymptote.
- $\Rightarrow$ arbitrarily large steady state WIP below capacity.

Flux vs. WIP in steady state for deterministic and stochastic mod



CONTENTS

- Part 1: Traffic flow vs. production systems
- Part 2: First principle models.
  - Simple deterministic first principle models $\longrightarrow$ hyperbolic conservation laws.
  - Stochasticity and kinetic models.
  - Mean field theory, long time averages and diffusive corrections.
- Part 3: Clearing distributions from observed data.
- Part 4:
  - The problem with diffusive corrections due to stochastic fluctuations. Hyperbolic relaxation models.

Stochasticity and Kinetic Models $_{26}$

Stochastic effects:

- Random arrivals (external influx), random breakdown of servers (medium).
- Random effects significantly influence the behavior.

Random server breakdowns $_{27}$

- Divide $[0,1]$ into K cells, each corresponding to one processor.

$$c(x,t) = \sum_k \chi_k(x)\mu_k(t)$$

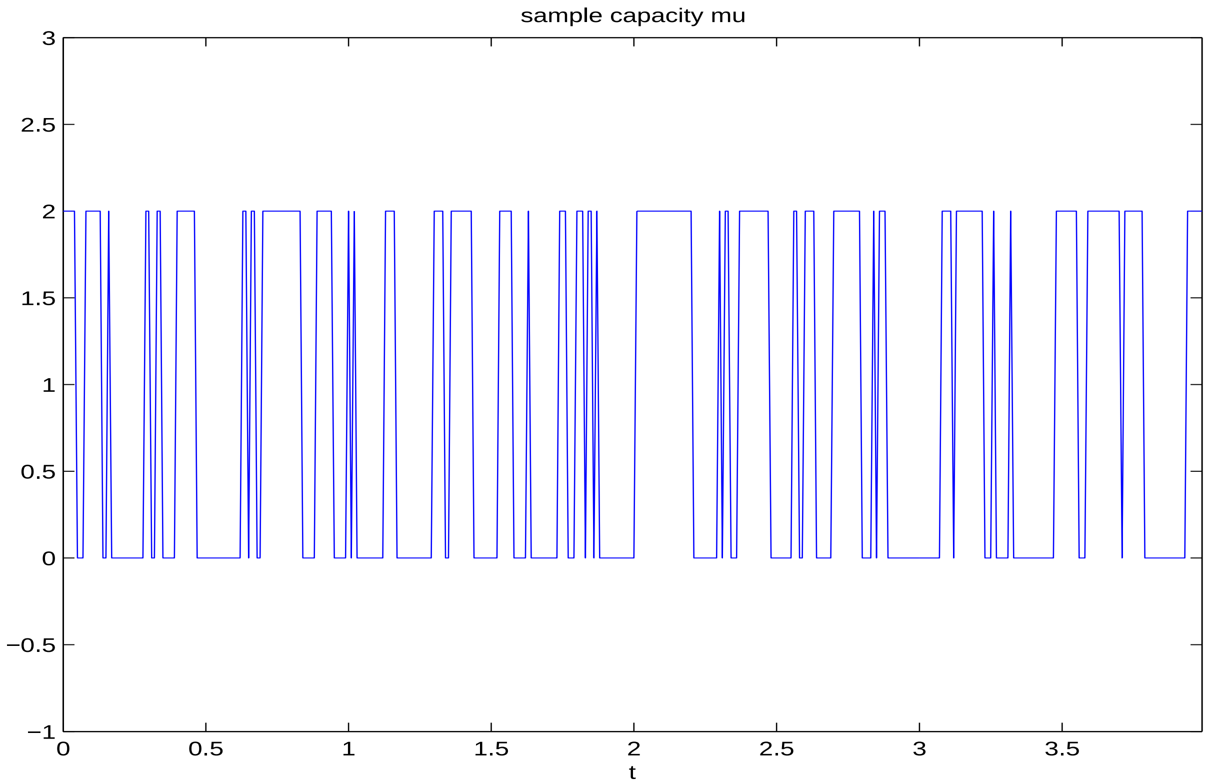
- The service rates $\mu_k(t)$ switch between $\mu_k = \mu_k^{up}(x)$ and $\mu_k = 0$ according to a Markov process.

$$\boxed{\mu_k(t+\Delta t) = \begin{pmatrix} \mu_k(t) & prob = 1 - \Delta t \frac{\omega_k(\mu_k)}{\varepsilon} \\ \mu_k^{up} - \mu_k(t) & prob = \Delta t \frac{\omega_k(\mu_k)}{\varepsilon} \end{pmatrix}}$$

$$\boxed{\mu_k(t+\Delta t) = \begin{pmatrix} \mu_k(t) & prob = 1 - \Delta t\frac{\omega_k(\mu_k)}{\varepsilon} \\ \mu_k^{up} - \mu_k(t) & prob = \Delta t\frac{\omega_k(\mu_k)}{\varepsilon} \end{pmatrix}}$$

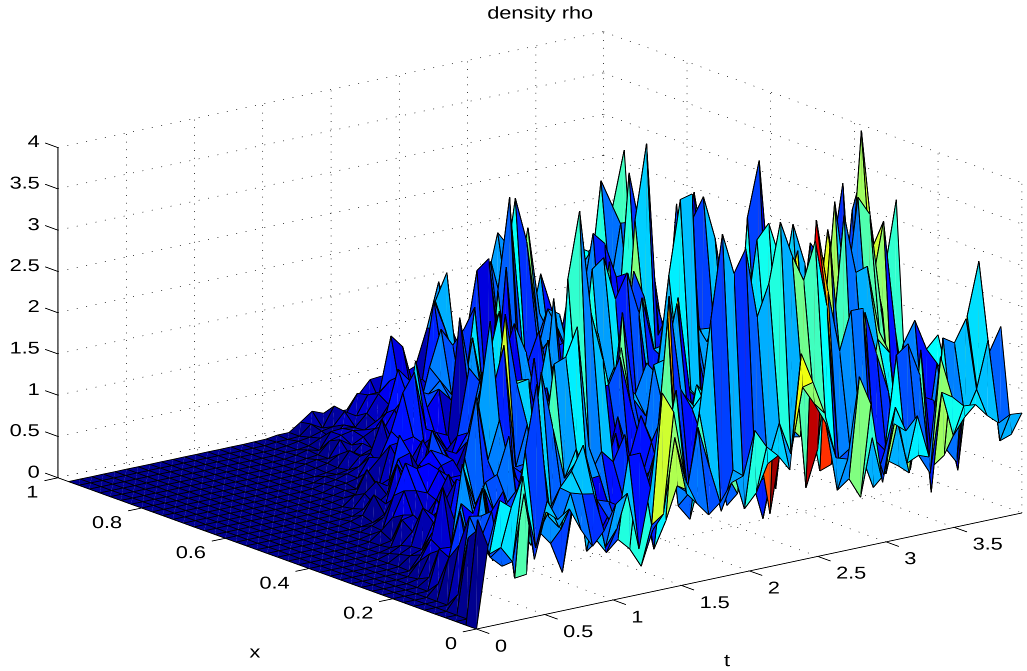
- $\frac{\omega_k}{\varepsilon}$ is the switching frequency $>> 1$.
- This is a Markov process, since the probability to change the state is independent of the time elapsed since the last change.

One realization of one μ_k



One realization of the density ρ

with flux $\phi = \min\{\mu, v\rho\}$ using a stochastic μ



Stochastic fluctuations in conservation laws

- Fluctuations cannot (in general) be included directly on the level of conservation laws.
- Include randomness in an underlying kinetic model.
- The kinetic model is very high dimensional (many body problem).
- Involves the motion of N parts in a medium governed by K random variables (the state of the processors).

General setting $_{33}$

- Consider N part(icle)s moving through state space $x = [0, 1]$.
- Part(icle) number n moves with velocity v_n and interacts with the medium (the production system)
- The production system is described by a state $C = (c_1, .., c_K)$.
- The state of the production system changes randomly.

$$x_n(t + \Delta t) = x_n(t) + \Delta t v_n(X, C), \quad d\mathcal{P}[q_k = p] = u_k(p, C)\, dp$$

$$c_k(t + \Delta t) = \begin{pmatrix} c_k(t) & \mathcal{P} = 1 - \Delta t \omega_k(C) \\ q_k & \mathcal{P} = \Delta t \omega_k(C) \end{pmatrix}$$

The many body problem $_{35}$

$$x_n(t+\Delta t) = x_n(t)+\Delta t v_n(X,C), \quad \mu_k(t+\Delta t) = \begin{pmatrix} c_k(t) & \mathcal{P} = 1 - \Delta \\ q_k & \mathcal{P} = \Delta t \end{pmatrix}$$

$$d\mathcal{P}[q_k = p] = u_k(p,C)\, dp$$

ω_k Define: $F(X,C,t)$ probability density that $(x_1,..,x_N)(t) = X$ and, at the same time, $(c_1,..,c_K)(t) = C$.

$\Rightarrow$

$$\partial_t F + \nabla_X \cdot (VF) = \frac{1}{\varepsilon} Q[F],$$

$$Q[F] = \sum_k \Big[\int u_k(c_k, C'_k)\omega_k(C'_k)F(X, C'_k, t)\, dc'_k - \omega_k(C)F \Big]$$

$$C'_k = (c_1, ..c'_k, ..c_K), \quad V = (v_1, .., v_N),$$

- This is a generalized version of the classical many body problem.
- $F(X, C, t)$ is a function of $N + K + 1$ variables $\Rightarrow$ Contains all the information about correlations up to any order, but not feasible for computations.

General Approach $_{38}$

- Mean field theory: replace many body equation for N particles by an effective one - body equation for a single particle.
- Large time limit for rapid interactions.

$$\partial_t F + \nabla_X \cdot (VF) = \frac{1}{\varepsilon} Q[F],$$

$$Q[F] = \sum_k [\int u_k(c_k, C'_k)\kappa_k(C'_k)F(X, C'_k, t)\, dc'_k - \kappa_k(C)F]$$

General Idea:

$$\partial_t F(x_1, .., x_N, t) + \nabla_X (V(X) F) = 0$$

Assume many identical and independent particles ($N >> 1$).

$$F(x_1, .., x_N, t) \approx \prod_1^N f(x_n, t)$$

$$\boxed{\partial_t f(x_1, t) + \nabla_{x_1}(\tilde{V}[f] f) = 0, \quad \tilde{V}[f] = \int V(X) \prod_{n=2}^N f(x_n, t)\, dx_2 .}$$

The Lagrangian picture:

$$V_n(X,t) = \min\{\frac{c(x_n)}{\rho(x_n)}, \sum_j \chi_j(x_n) v_0(x_n)\}$$

Specific volume:

$$\frac{1}{\rho(x_n)} = \min\{x_j - x_n : \; x_j > x_n\}$$

Modification:

$$\partial_t F(x_1, .., x_N, t) + \nabla_X (V(X)F) = 0$$

Particles independent of each other but not of the medium. $\Rightarrow$

statistical independence for the conditional probability.

$$\boxed{F(X, C, t) = \frac{F_0(X,C,t)}{G(C,t)}, \quad G(C, t) = \int G(X', C, t)\, dX'}$$

$$F_0(X, C, t) = \prod_n f(x_n, C, t)$$

$$\boxed{\partial_t f(x_1, C, t) + \frac{1}{G}\nabla_{x_1}(\tilde{V}[f]Gf(x, C)) = 0}$$

Theorem (Degond, CR '06)

For $N \to \infty$ the mean field velocity $\tilde{V}$ is given by

$$\boxed{\tilde{V}[f](x, C, t) = \frac{\mu(x,C)}{G}[1 - \exp(-\frac{v_0(x) f G}{\mu(x,C)})],}$$

$$\mu(x, C) = \sum_k \chi_k(x) c_k$$

$$\boxed{\Rightarrow \partial_t(fG) + \nabla_x(\tilde{V} fG) = \frac{1}{\varepsilon} Q[fG]}$$

Large time asymptotics $_{43}$

$$\boxed{\partial_t(fG) + \nabla_x(\tilde{V} fG) = \frac{1}{\varepsilon} Q[fG]}$$

($\varepsilon \longrightarrow 0$) to derive an equation for

$$\rho(x,t) = \int f(x,C,t) G(C,t)\, dC$$

Insert: The Chapman Enskog Expansion $_{44}$

$$\boxed{\partial_t(fG) + \nabla_x(\tilde{V} fG) = \tfrac{1}{\varepsilon} Q[fG]}$$

set $g(x, C, t) = fG$

$$\partial_t g(x, C, t) + \partial_x[\tilde{V}(x, C)g] = \frac{1}{\varepsilon} Q[g]$$

Separate the time scales by projecting onto the kernel of Q and its orthogonal com[lement.

$$Q[g] = 0 \iff g(x, C, t) = \rho(x, t) G_0(x, C), \quad \int G_0 \, dC = 1, \, \forall x$$

$$\boxed{Pg(x, C, t) = \rho(x, t) G_0(x, C), \quad \int (I - P) g(x, C, t) \, dC = 0}$$

$$\boxed{Pg(x,C,t) = \rho(x,t)G_0(x,C), \quad \int (I-P)g(x,C,t)\,dC = 0}$$

$$\Rightarrow PQ = QP = 0$$

Separate the time scales:

$$g(x,C,t) = g_0 + \varepsilon g_1, \quad g_0 = Pg, \quad \varepsilon g_1 = (I-P)g$$

$$\boxed{(1)\ g_0 + P\partial_x[\tilde{V}(g_0+\varepsilon g_1)] = 0,}$$

$$\boxed{(2)\ \varepsilon\partial_t g_1 + (I-P)\partial_x[\tilde{V}(g_0+\varepsilon g_1)] = Q[g_1]}$$

$$(1)\ g_0 + P\partial_x[\tilde{V}(g_0 + \varepsilon g_1)] = 0,$$

$$(2)\ \varepsilon\partial_t g_1 + (I - P)\partial_x[\tilde{V}(g_0 + \varepsilon g_1)] = Q[g_1]$$

(1) gives the macroscopic equation on the slow time scale.

$$g_0 = \rho G_0 \Rightarrow \partial_t \rho + \partial_x[\int \tilde{V}(g_0 + \varepsilon g_1)\, dC] = 0$$

(2) gives the infinite dimensional remainder on the fast time scale.

$$(2)\ \varepsilon\partial_t g_1 + (I-P)\partial_x[\tilde{V}(g_0+\varepsilon g_1)] = Q[g_1]$$

Solve (2) asymptotically for $\varepsilon \longrightarrow 0$

$$(2) \rightarrow (I-P)\partial_x[\tilde{V} g_0] = Q[g_1]$$

This gives the macroscopic system

$$\boxed{\partial_t \rho(x,t) + \partial_x[\int \tilde{V}(\rho G_0 + \varepsilon g_1)\, dC] = 0}$$

$$\boxed{(I-P)\partial_x[\tilde{V}\rho G_0] = Q[g_1]}$$

The term $\varepsilon \int \tilde{V} g_1\, dC$ gives a small diffusive correction to the hyperbolic equation.

The zero order term $_{48}$

$$g = fG(x, C, t) = \rho(x, t) G_0(C), \quad Q[G_0] = 0$$

$\Rightarrow \rho(x, t)$ satisfies the expectation of the LHS under the equilibrium measure G_0.

$$\boxed{\partial_t \rho + \nabla_x(\tilde{v}\rho) = 0, \quad \tilde{v} = \frac{a\mu^{up}}{\rho}[1 - \exp(-\frac{v_0(x)\rho}{\mu^{up}(x)})]}$$

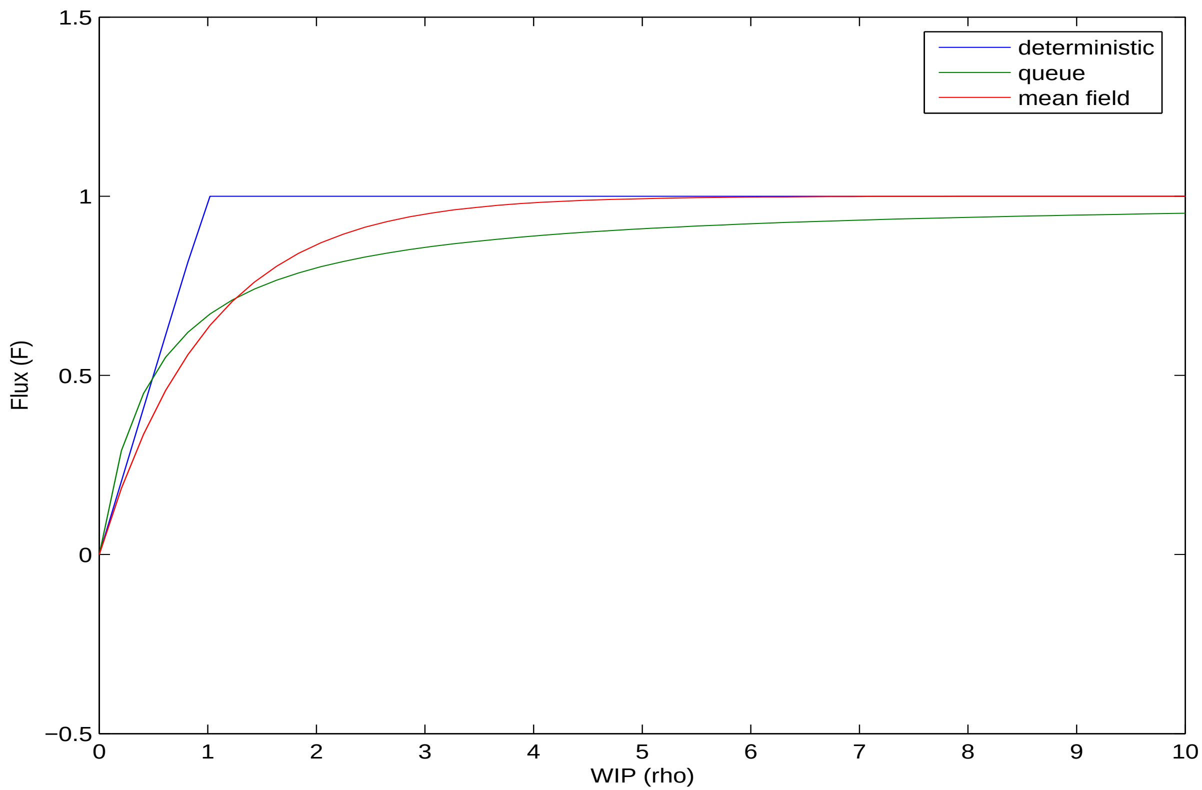
a: availability (=$\sum_k \chi_k(x) \frac{\omega_k(0)}{\omega_k(0)+\omega_k(\mu^{up})}$)

Interpretation $_{49}$

$$\tilde{v} = \frac{a\mu^{up}}{\rho}[1 - \exp(-\frac{v_0(x)\rho}{\mu^{up}(x)})]$$

- Compare ϕ to the expectation flux ϕ_E
- μ is replaced by the 'on' capacity μ^{up}.
- The function $\min\{\mu, v\rho\}$ is replaced by $\mu[1 - e^{-\frac{v\rho}{\mu}}]$, which has the same asymptotic properties for $\rho \longrightarrow 0$ and $\rho \longrightarrow \infty$.
- The whole flux ϕ is multiplied by the factor $\tau_{up}^{av} = \frac{\langle\tau_{up}\rangle}{\langle\tau_{up}\rangle + \langle\tau_{down}\rangle}$.

Flux diagram $_{51}$



deterministic vs. $M/M/1$ queue vs. long time average mean field theory

Issues $_{52}$

- Analysis and model based on Markov processes.
- Not appropriate for repair and scheduled maintenance.
- The zero order term only uses information about the expectations of the process.
- Stochastic fluctuations influence the higher order terms in the expansion.

Fluctuations

The $O(\varepsilon)$ term in the expansion:

$$g(x,C,t) = fG = \rho(x,t)G_0(C) + \varepsilon g_1(x,C,t)$$

$$\boxed{\partial_t \rho(t,x) + \partial_x\{ac_{up}(x)[1 - \exp(-\frac{v_0\rho}{c_{up}})] - a\varepsilon\mathcal{V}^2\partial_x\rho\} = 0}$$

$a(x)$: availability $= \frac{\langle T_{up}\rangle}{\langle T_{up}\rangle + \langle T_{down}\rangle}$, $(\langle T\rangle = \frac{1}{\omega})$

$\mathcal{V} = \frac{\sigma}{\langle T\rangle}$: variation coefficient $(= 1$ for a Markov process).

- Since we consider many rapidly switching processors, stochastic fluctuations average out, and appear only as an $O(\varepsilon)$ correction.

Non - Markov processes [56]

Problem:

$$\boxed{c(x,t+\Delta t) = \begin{pmatrix} c(x,t) & prob = 1 - \Delta t\omega(x,c) \\ c^{up}(x) - c(x,t) & prob = \Delta t\omega(x,c) \end{pmatrix}}$$

- No memory of how long the processor has been running or down.
- T_{up}, T_{down} (times between switches) distributed according to exponential distributions $\Rightarrow \mathcal{V} = 1$.

$$\boxed{\mathcal{P}[T_{up/down} = \tau] = \frac{1}{\omega(\mu^{up}/0)} \exp[-\omega\tau]}$$

- Analogy to 'intelligent particles' with memory (drivers).
- Model an arbitrary stochastic process, by enlarging the state space.

A different game $_{59}$

τ: time elapsed since the last change of state of the processor.

$$\boxed{c(x,t+\Delta t) = \begin{pmatrix} c(x,t) & prob = 1-\Delta t\omega(x,c,\tau) \\ c^{up}(x) - c(x,t) & prob = \Delta t\omega(x,c,\tau) \end{pmatrix}}$$

$$\boxed{\tau(x,t+\Delta t) = \begin{pmatrix} \tau(x,t)+\Delta t & prob = 1-\Delta t\omega(x,c,\tau) \\ 0 & prob = \Delta t\omega(x,c,\tau) \end{pmatrix}}$$

- Larsen ('07) (Radiative transfer models in clouds)

Lemma

$u(x, c, s)\ ds$: distribution of the time between scattering events.

$$\boxed{u(x, c, s)\ ds = d\mathcal{P}[\tau = s] = \omega(x, c, s) \exp[-\int_0^s \omega(x, c, q)\ dq]\ ds}$$

$\Rightarrow$ given any kind of probability distribution $u(x, c, \tau)\ d\tau$ of up / down times, we define a (τ dependent) frequency ω

$$\boxed{\omega(x, c, \tau) = \frac{u(x,c,\tau)}{1 - \int_0^\tau u(x,c,s)\ ds}}$$

$$u(x,c,s)\,ds = d\mathcal{P}[\tau = s] = \omega(x,c,s)\exp[-\int_0^s \omega(x,c,q)\,dq]\,ds$$

- This is a generalization of the Markov process, since, for $\omega(x,c,\tau) = \omega(x,c)$ we have

$$u(x,c,\tau) = \omega(x,c)\exp[-\tau\omega(x,c)]$$

- Using this trick, everything stays the same, except for the fact that the conditional probability (in the mean field assumption) satisfies

$$\boxed{\partial_t f(t,x,C,\overrightarrow{\tau}) + \partial_x \Phi = \frac{1}{\varepsilon} Q[f],}$$

$$\boxed{Q[f] =}$$

$$\boxed{-\nabla_{\overrightarrow{\tau}} f + \sum_j \delta(\tau_j) \int K(x,C,C')\Omega f(t,x,C',\tau')\, dC'\tau' - \Omega(x,C,\tau)}$$

CE for the modified process $_{64}$

Large time asymptotics (Chapman - Enskog) gives the same result

$$\boxed{\partial_t \rho(t,x) + \partial_x \{ a c_{up}(x) [1 - \exp(-\frac{v_0 \rho}{c_{up}})] - a\varepsilon D(\rho) \mathcal{V}^2 \partial_x \rho \}}$$

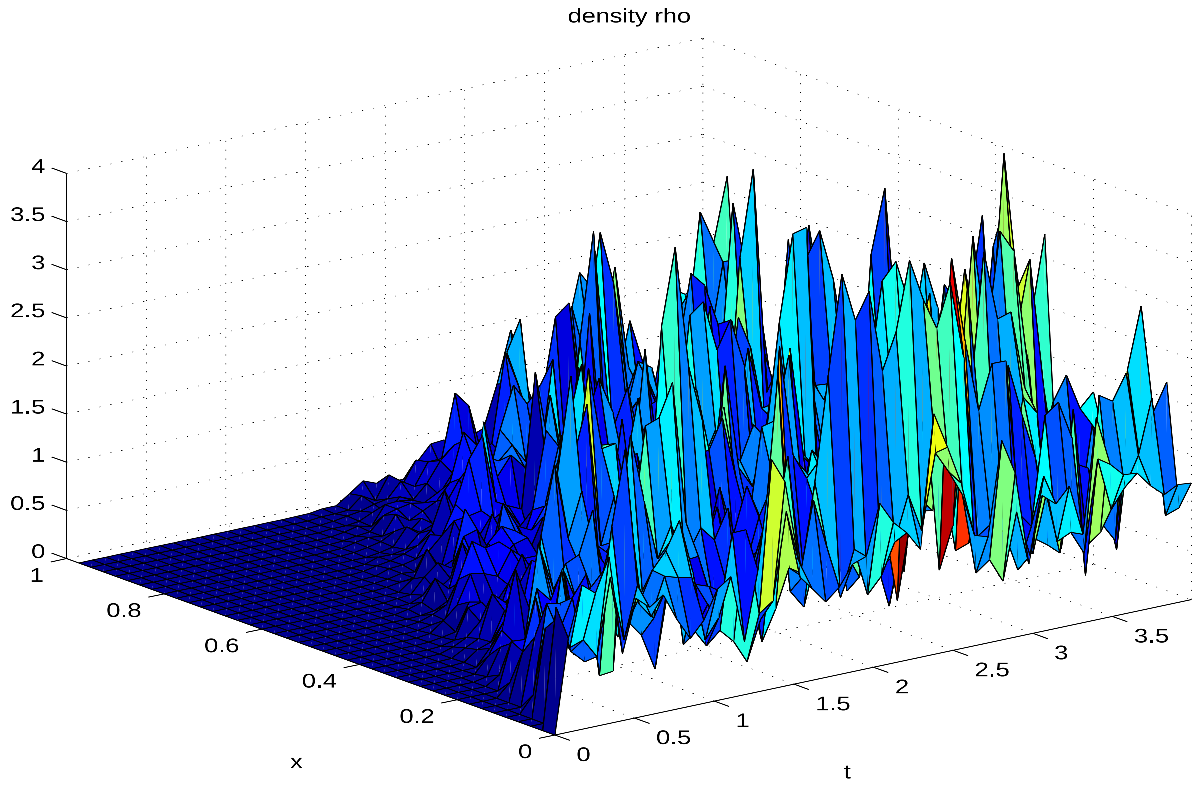
- $\mathcal{V}$: Variation coefficient for the up and down times of the general underlying switching process.

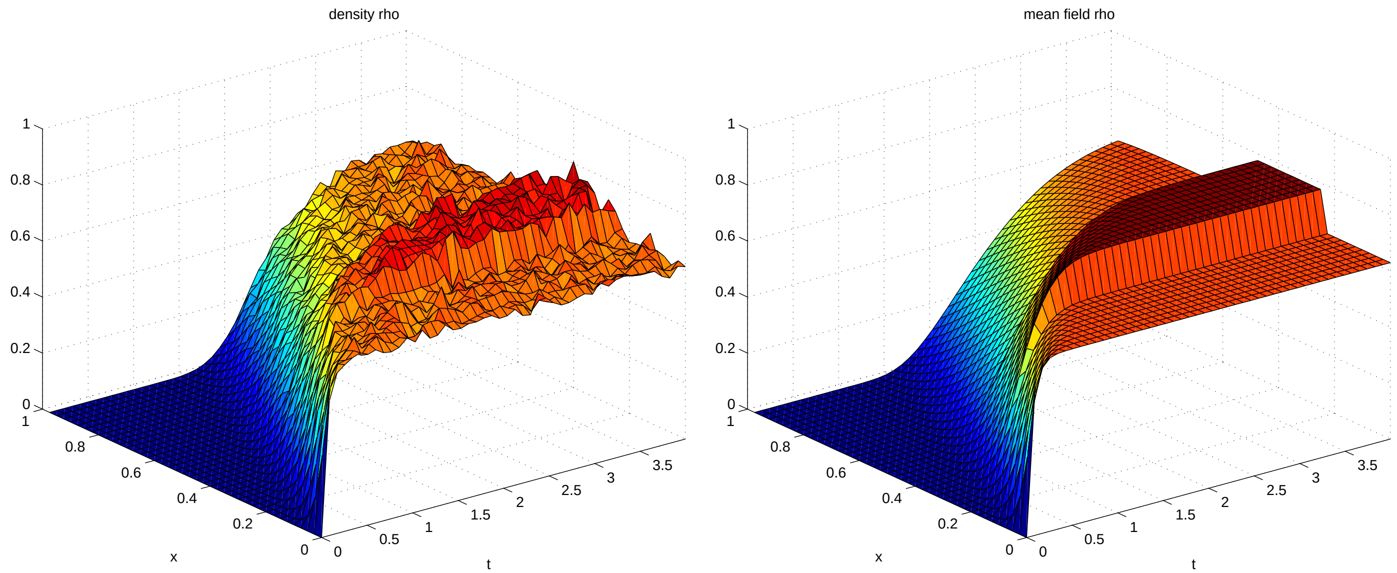
Numerical Verification - Experiment 1:

Quantitative comparison of steady states

- 40 processors with a peak capacity $c^{up} = 2$ and $v = 1$.
- A bottleneck in processors $11 - 20$ (peak capacity $c^{up} = 1$)
- They are on half of the time
  $\langle T_{up} \rangle = \frac{1}{\omega(2)} = \langle T_{down} \rangle = \frac{1}{\omega(0)} = \frac{1}{20}$
- Solve with the flux $\phi = \min\{c, v\rho\}$ for 200 realizations of μ and average.
- Constant influx $\phi(x = 0) = 0.25$
- Compare to the solution for $\langle \rho \rangle$ of the equation or the mean and variance.

One realization



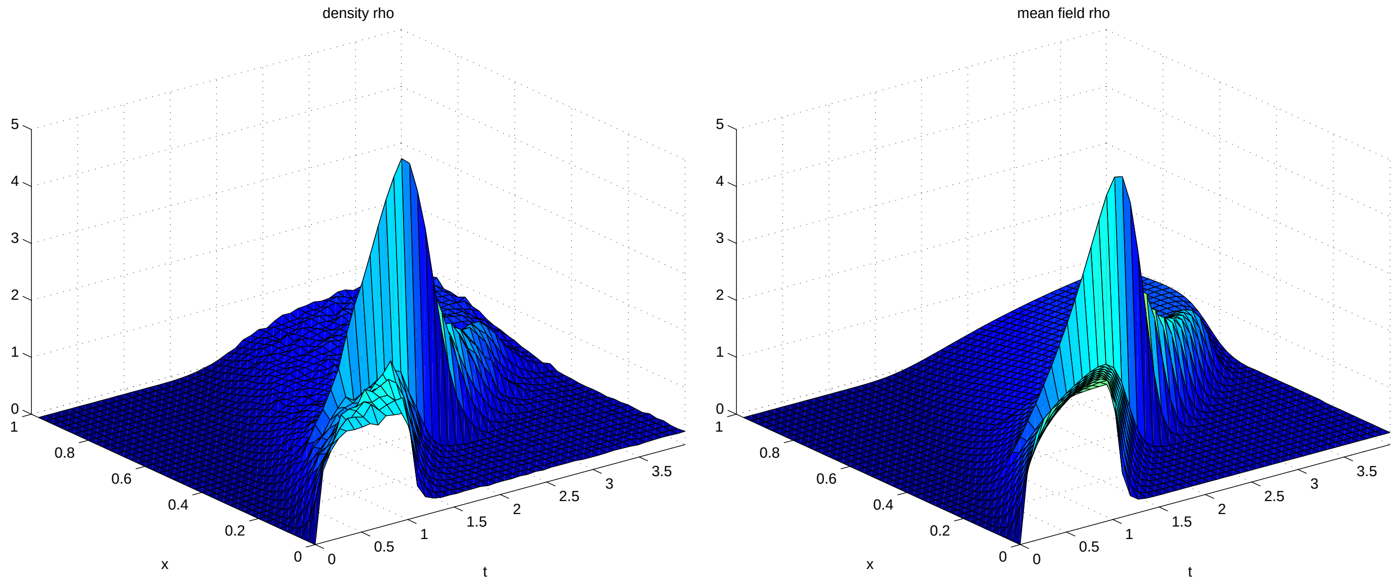


Comparison for constant influx $F(x = 0) = 0.25$

Experiment 2:

The transient response

Influx temporarily above the bottleneck capacity $c_{btlnck} = 0.5$; average over 500 realizations.



CONTENTS

Part 3: Clearing distributions from observed data $_{75}$

- Observe a specific system (experimentally) for a period of time.
- Compute a (large) table:

  $a_{mn}, m = 1 : M,\ n = 1 : N$: time part number n has completed stage number m of the process.
- Extract transport coefficients directly from observed data.
- Build a kinetic model using numerical distributions.
- Construct a macroscopic model via long time averages, using means and variances of the observed distributions.

A kinetic model $_{77}$

$x = \xi_n(t)$ stage of part number n at time t. v_n: velocity

$$\xi_n(t + \Delta t) = \xi_n(t) + \Delta t v_n(t)$$

v_n changes randomly, according to a distribution extracted from the observed data.

$$v_n(t + \Delta t) = (1 - r)v_n(t) + r\eta_n, \quad \mathcal{P}[r = 1] = \omega \Delta t$$

$$d\mathcal{P}[\eta_n = v] = U(x, t, v)\, dv$$

The kinetic equation:

$$\boxed{\partial_t f(x,v,t) + \partial_x [v f] = Q[f],}$$

$$\boxed{Q[f] = U(x,v) \int \omega(v') f(x,v',t)\, dv' - \omega f}$$

ω: scattering frequency

Make the distribution U dependent on some macroscopic functional of the part density

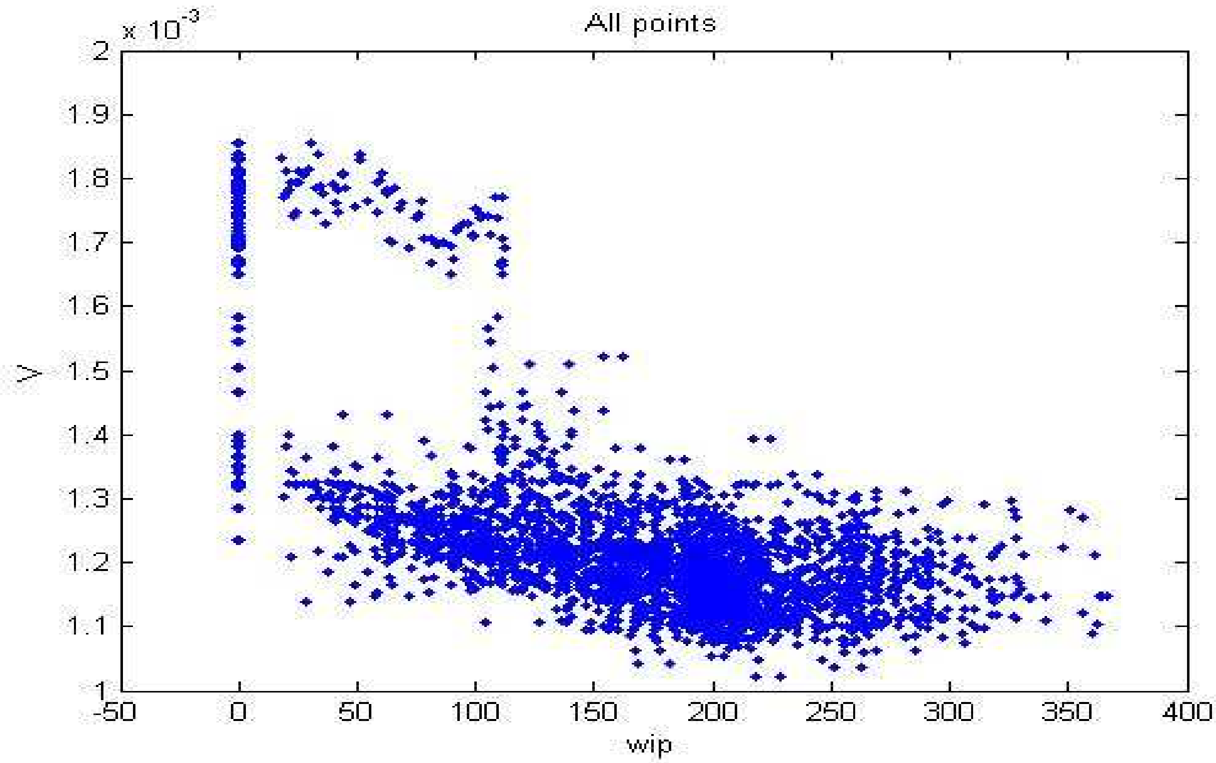
$$\boxed{U(x,v) = U_S(x,v,s),}$$

$$S = \begin{pmatrix} \int f(x,v,t)\, dv \\ \int f(x,v,t)\, dxv \\ \int_x^1 dx \ \int dv\, f(x,v,t) \end{pmatrix}$$

Instead of a clearing function (a flux or a group velocity) $v(\rho) = \frac{F(\rho)}{\rho}$ we have a velocity distribution $U_S(x, v, s)\, dv$, given in terms of a conditional probability.

$$\boxed{U_S(x, v, s)\, dv = d\mathcal{P}[\eta = v | S(x, t) = s]}$$

Velocity distribution at stage 38 of the process (depending on the local WIP)



The update frequency $\omega(v)$ $_{82}$

Mean free path:

$M >> 1$ stages $\Rightarrow \frac{v}{\omega} = \frac{1}{M} \Rightarrow \omega(v) = Mv,\ M >> 1$

The kinetic model:

$$\boxed{\partial_t f(x,v,t) + \partial_x [vf] = \frac{1}{\varepsilon} Q[f],}$$

$$\boxed{Q[f] = U(x,v,S) \int \omega(v') f(x,v',t)\, dv' - \omega f,\ \varepsilon = \frac{1}{M} << 1}$$

$$S = \begin{pmatrix} \int f(x,v,t)\, dv & or \\ \int f(x,v,t)\, dxv & or \\ \int_x^1 dx\ \int dv\ f(x,v,t) & \end{pmatrix}$$

Chapman - Enskog for large time scales:

Split again the slow and fast time scales to obtain diffusion equation for the macroscopic density.

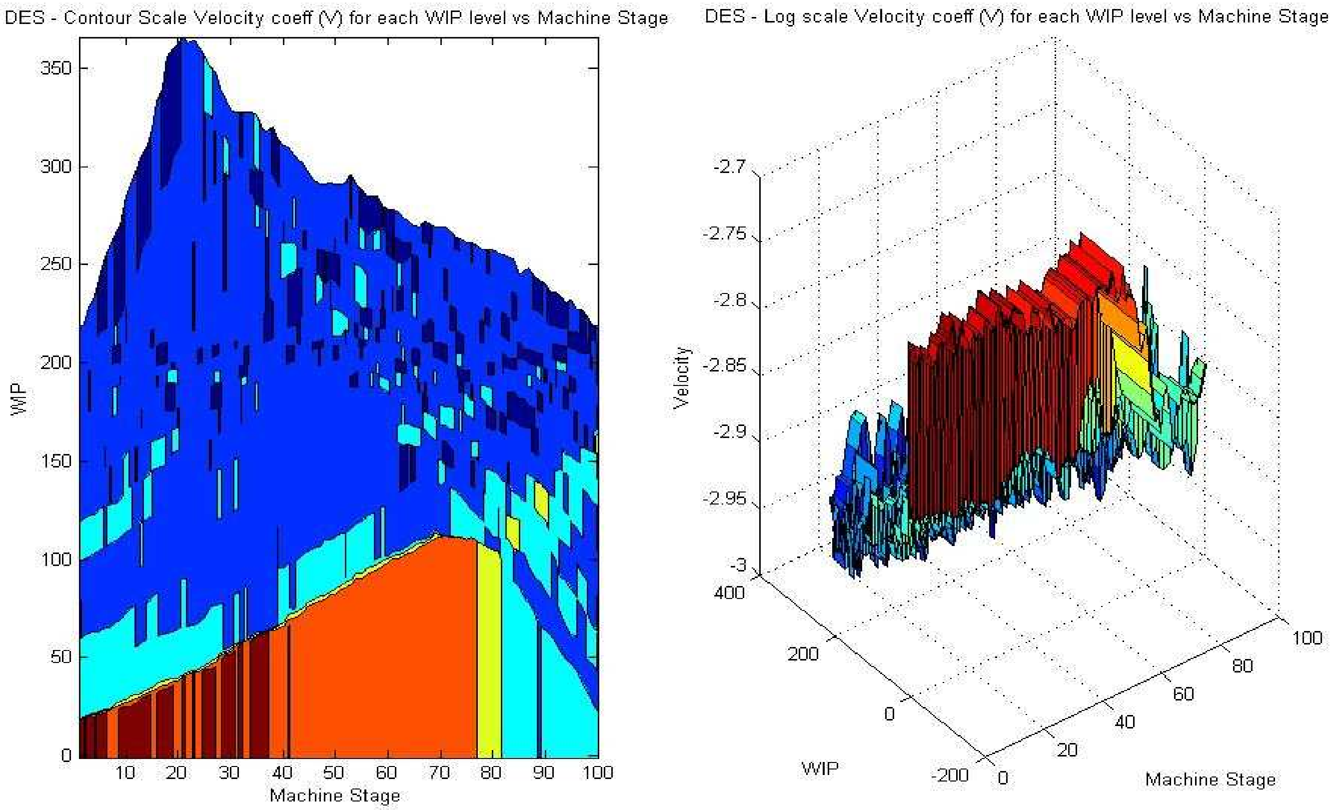
$$f = f_0 + \varepsilon f_1, \quad f_0 = Pf, \quad \varepsilon f_1 = (I - P)f$$

$$Pf = \frac{\rho U(x,v,S)}{\omega}, \quad \rho = \int f\, dv$$

$$\boxed{\partial_t \rho(x,t) + \partial_x[\langle V\rangle \rho - \varepsilon D \partial_x \rho] = 0, \quad D = \frac{\sigma(U)}{\langle u\rangle}}$$

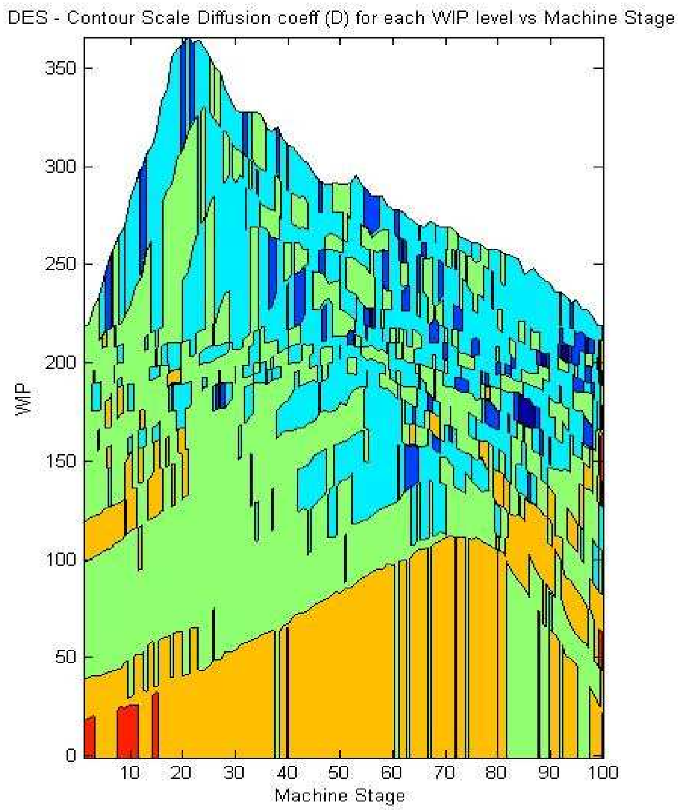
The amount of stochasticity in the system is given by the variation coefficient $\frac{\sigma(U)}{\langle U >}$.

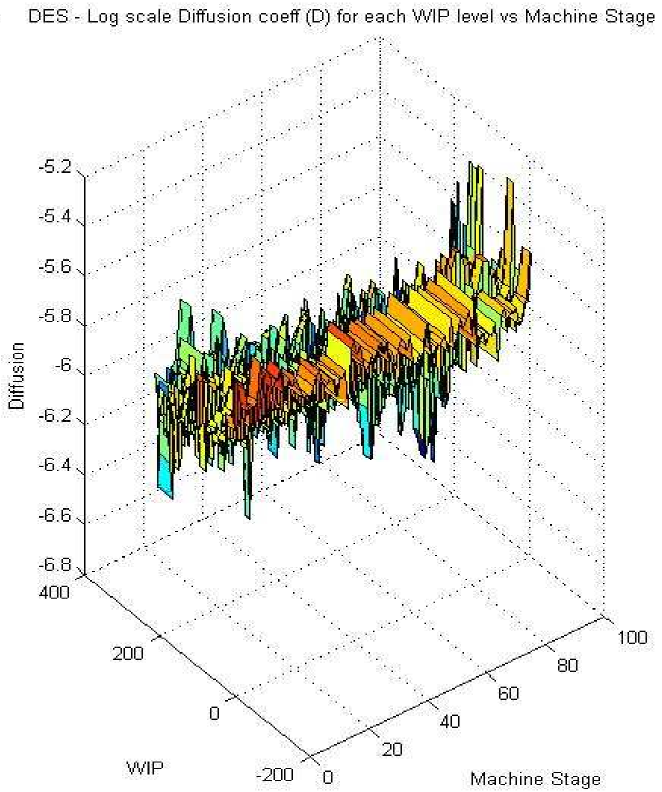
Mean velocity as a function of total WIP



1

Diffusion coefficient as function of total WIP

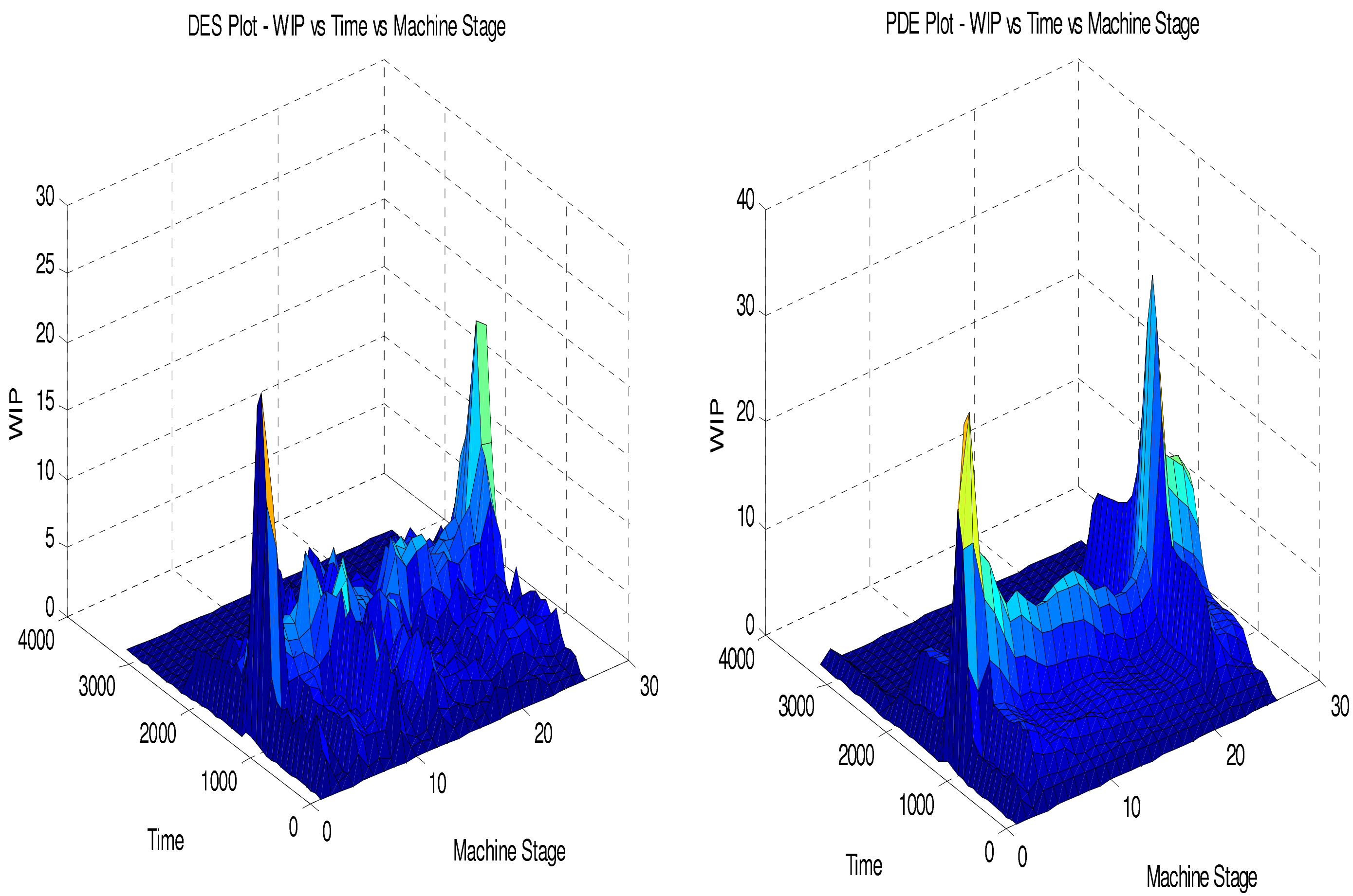




1

Some Results $_{87}$

- Toy factory - Comparison (WIP) to discrete event simulations
- 26 processing steps, 200 machines, FIFO



Left: DES: (60,000 lots, 100 realizations), Right: CL

CONTENTS

Diffusion vs. Hyperbolicity $_{91}$

The basic problem:

- A diffusion equation (as a result of Chapman - Enskog) propagates information in both directions.
- Parts (or drivers in traffic flow) do not react to what is happening behind them.
- This is an artifact of the Chapman - Enskog procedure which transforms diffusion (arising from the random fluctuations in the flow) in velocity into spatial diffusion in a macroscopic limit.
- General problem for directional flows and fluctuations.

- In practice, the equation

$$\boxed{\partial_t \rho(t,x) + \partial_x\{a c_{up}(x)[1-\exp(-\frac{v_0 \rho}{c_{up}})] - a\varepsilon D(\rho)\mathcal{V}^2 \partial_x \rho\}}$$

needs a boundary condition at $x=1$ and there is no 'physics' to determine this condition.

Hyperbolic Relaxation Models - Basic Idea $_{96}$

- Solve the kinetic equation by a moment closure, taking additional (not conserved) moments.
- $\Longrightarrow$ a hyperbolic system, still containing ε.
- Close the moment hierarchy by an ansatz, such that $\varepsilon \longrightarrow 0$ asymptotics on the macroscopic level would reproduce the diffusion picture.
- Don't do it. Use the hyperbolic model instead.
- Natalini, Jin, Slemrod ('95): Regularization of the Burnett and super - Burnett equations.

The Chapman - Enskog expansion revisited:

$$\partial_t g + \partial_x(\tilde{V} g) = \frac{1}{\varepsilon} Q[g]$$

$$g = g_0 + \varepsilon g_1 \quad g_0 = Pg = \rho(x,t) G_0(x,C), \quad \varepsilon g_1 = (I - P) g$$

$$(1)\ \partial_t(\rho G_0) + \partial_x[P\tilde{V}(\rho G_0 + \varepsilon g_1)] = 0$$

$$(2)\ \varepsilon \partial_t g_1 + (I - P)\partial_x[\tilde{V}(\rho G_0 + \varepsilon g_1)] = Q[g_1]$$

In standard C.E. equation (2) is solved asymptotically, replacing it by

$$(2)\ (I-P)\partial_x[\tilde{V}(\rho G_0+\varepsilon g_1)] = Q[g_1]$$

$$(1)\ \partial_t(\rho G_0)+\partial_x[P\tilde{V}(\rho G_0+\varepsilon g_1)] = 0$$

which gives a diffusion term in the equation for ρ

$$\boxed{\partial_t\rho+\partial_x[\rho\langle\tilde{V}G_0\rangle+\varepsilon\langle g_1\rangle] = 0}$$

Alternative:

- Solve the original equation (2)

$$(2)\ \varepsilon\partial_t g_1 + (I-P)\partial_x[\tilde{V}(\rho G_0 + \varepsilon g_1)] = Q[g_1]$$

  by a moment closure and close with an ansatz arising from the C.E. asymptotics.

- This gives a system which still contains the fast time scale ($O(\frac{1}{\varepsilon})$ terms).

- The system should be hyperbolic and (formally) reduce to the classical C.E. diffusion equation for $\varepsilon \to 0$.

One more moment $_{98}$

$$\boxed{\partial_t \rho + \partial_x (u\rho) = 0, \quad \partial_t (u\rho) + \partial_x [u^2 \rho + P\rho] = \frac{\rho}{\varepsilon}(u_0 - u)}$$

$$u\rho = \int \tilde{V}(\rho G_0 + \varepsilon g_1)\, dC, \quad u^2\rho + P\rho = \int \tilde{V}^2 (\rho G_0 + \varepsilon g_1)\, dC$$

P: pressure (from the closure).

▶ Close with the asymptotic form given by Chapman - Enskog:

$$\rho u^2 + \rho P = \int \tilde{V}^2 (\rho G_0 + \varepsilon g_1)(x, C, t)\, dC$$

g_1 taken from the classical Chapman - Enskog procedure.

$$\tilde{V} \rho G_0 = Q[g_1]$$

Theorems $_{100}$

- Asymptotics on the hyperbolic level:

$$u(x,t) = u_0 - \frac{\varepsilon}{\rho}\partial_x[P\rho]$$

gives up to order ε^2 the same equation as the standard Chapman Enskog expansion.

- The characteristic speeds of the second order system

$$\boxed{\partial_t\rho + \partial_x(u\rho) = 0, \quad \partial_t(u\rho) + \partial_x[u^2\rho + P\rho] = \frac{\rho}{\varepsilon}(u_0 - u)}$$

are non - negative for $\varepsilon << 1$.

Remark

$$\boxed{\partial_t \rho + \partial_x (u\rho) = 0, \quad \partial_t (u\rho) + \partial_x [u^2 \rho + P\rho] = \frac{\rho}{\varepsilon}(u_0 - u)}$$

- The numerical solution of the relaxation model causes no additional difficulties, since the (local) relaxation term can be discretized implicitly.
- Since the characteristics all point to the right, there is no boundary condition needed at $x = L$.
- The boundary condition for the energy $u^2\rho + P\rho$ has to be computed from the equilibrium energy in the standard C.E. expansion.

Issues

- The sign of the phase speeds has to be proven individually for a certain velocity profile $\tilde{V}$ and a certain collision operator Q.
- In the nonlinear case $\tilde{V} = \tilde{V}(\rho)$ we can guarantee the sign only in the limit $\varepsilon \to 0$.